St Mary's CE High School

A Level Mathematics

Y11 into Y12 Transition Work

Summer 2020

Name:

Deadline: First Further Maths lesson in September

NOTE: There are ***two*** tasks to complete

Task One: Academic

The following are all topics you should be confident with.

Please work through these tasks to help prepare for the test you will be sitting the second week back in September.

It would be useful to keep the notes/workings you do so we can tackle any issues in September.

Topic List	Task	Studied?	Understood?
Surds	**Worksheet A**		
Vectors	**Worksheet B**		
Calculus	**Worksheet C**		
Matrices	**Worksheet D**		

Please Note:
You do not have to do every question – you need to do enough so you understand every type of question.

We have provided the answers so that you can mark as you go, but you must ensure you have thoroughly attempted a question before looking at the answers.

Task Two: Research

The most common complaint that students have about mathematics and algebra in particular is "When am I ever going to use this?". You are going to defend the various topics that you will be studying throughout this year. You need to:

1. *Research the history of your topic, including prominent mathematicians involved with that topic*
   The history of your topic can be gathered from several resources - textbooks, encyclopaedias or from sources of math history on the internet.

2. *Show how your topic is used in the "real world"*
   Real world uses can be gathered from several sources. A starting place is your text where you can gather occupations that use your topic.
   Create several (4-6) sample problems that would demonstrate real world use and answer them.

3. *Find ways that you might use this topic today*
   Find out about the real life usage of your topic. You could write to mathematicians to find answers!

4. *Evaluate your project*
   Even though there are some topics in maths that you might find more difficult than others, your responsibility is to not let past impressions influence you.
   You are to evaluate your report that "defends" your topic to students that ask 'When I am ever going to use this?'.

Topics

The following is a list of topics from which to choose:

- Complex numbers
- Matrices
- Calculus

You may choose something else but it must be covered in the A level specification.

Assessment

Your project will be assessed on the following criteria:

- Are all three parts of the brief in the project?
- Are the parts clear and well written?
- Is your information well supported?
- Is your information informative and persuasive?

Resources

Dictionary - http://www.m-w.com/dictionary.htm

The Math Forum - http://forum.swarthmore.edu/

Math History - http://www.aloha.net/~bry/teaching/math.html

History of Math - http://www.tc.cornell.edu/Edu/MathSciGateway/math.html

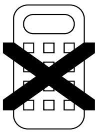
This is MegaMathematics - http://www.c3.lanl.gov/mega-math/menu.html

Ask Dr. Math - http://forum.swarthmore.edu/dr.math/drmath.high.html

Math Images - http://archives.math.utk.edu/images.html

Summer 2020 Further Maths Transition work

Worksheet A – Surds (and answers)



Surds.

Multiplying Surds.

A. Express these surds in the form $a\sqrt{b}$.

1). $\sqrt{8}$	2). $\sqrt{27}$	3). $\sqrt{20}$	4). $\sqrt{32}$	5). $\sqrt{80}$
6). $\sqrt{44}$	7). $\sqrt{75}$	8). $\sqrt{72}$	9). $\sqrt{45}$	10). $\sqrt{108}$
11). $\sqrt{28}$	12). $\sqrt{125}$	13). $\sqrt{245}$	14). $\sqrt{192}$	15). $\sqrt{405}$
16). $\sqrt{112}$	17). $\sqrt{63}$	18). $\sqrt{180}$	19). $\sqrt{99}$	20). $\sqrt{48}$

B. Express each of the following as the square root of a single number.

1). $3\sqrt{2}$	2). $2\sqrt{5}$	3). $6\sqrt{2}$	4). $4\sqrt{5}$	5). $3\sqrt{3}$
6). $2\sqrt{3}$	7). $5\sqrt{5}$	8). $7\sqrt{2}$	9). $6\sqrt{3}$	10). $5\sqrt{11}$
11). $9\sqrt{3}$	12). $10\sqrt{7}$	13). $8\sqrt{5}$	14). $11\sqrt{3}$	15). $15\sqrt{6}$

C. Work out the following. Leave the answer in surd form where appropriate.

1). $\sqrt{3} \times \sqrt{6}$	2). $\sqrt{6} \times \sqrt{2}$	3). $\sqrt{10} \times \sqrt{5}$	4). $\sqrt{8} \times \sqrt{5}$	5). $\sqrt{10} \times \sqrt{2}$
6). $\sqrt{3} \times \sqrt{3}$	7). $\sqrt{2} \times \sqrt{8}$	8). $\sqrt{14} \times \sqrt{2}$	9). $\sqrt{2} \times \sqrt{9}$	10). $\sqrt{5} \times \sqrt{15}$
11). $\sqrt{3} \times \sqrt{8}$	12). $\sqrt{5} \times \sqrt{5}$	13). $\sqrt{2} \times \sqrt{18}$	14). $\sqrt{6} \times \sqrt{6}$	15). $\sqrt{5} \times \sqrt{30}$

D. Surds in the form $a\sqrt{b}$ can be multiplied.

E.g. $3\sqrt{3} \times 4\sqrt{2} = 12\sqrt{6}$

Work out the following. Simplify where possible.

1). $2\sqrt{7} \times 4\sqrt{3}$	2). $2\sqrt{5} \times 3\sqrt{2}$	3). $2\sqrt{3} \times 3\sqrt{3}$	4). $5\sqrt{3} \times 7\sqrt{2}$
5). $4\sqrt{2} \times 5\sqrt{2}$	6). $2\sqrt{3} \times 6\sqrt{5}$	7). $2\sqrt{8} \times 5\sqrt{5}$	8). $2\sqrt{3} \times 3\sqrt{8}$
9). $2\sqrt{2} \times 3\sqrt{8}$	10). $2\sqrt{32} \times 3\sqrt{2}$	11). $5\sqrt{6} \times 2\sqrt{3}$	12). $2\sqrt{7} \times 3\sqrt{3}$
13). $5\sqrt{5} \times 4\sqrt{4}$	14). $4\sqrt{24} \times 2\sqrt{3}$	15). $5\sqrt{3} \times 4\sqrt{21}$	16). $3\sqrt{5} \times 4\sqrt{24}$
17). $4\sqrt{8} \times 3\sqrt{5}$	18). $3\sqrt{2} \times 8\sqrt{40}$	19). $4\sqrt{18} \times 2\sqrt{5}$	20). $2\sqrt{6} \times 5\sqrt{42}$

Dividing Surds.

A. Work out the following. Simplify where possible.

1). $\sqrt{10} \div \sqrt{2}$	2). $\sqrt{20} \div \sqrt{5}$	3). $\sqrt{150} \div \sqrt{3}$	4). $\sqrt{90} \div \sqrt{2}$	5). $\sqrt{72} \div \sqrt{6}$
6). $\sqrt{42} \div \sqrt{7}$	7). $\sqrt{55} \div \sqrt{5}$	8). $\sqrt{144} \div \sqrt{8}$	9). $\sqrt{48} \div \sqrt{6}$	10). $\sqrt{126} \div \sqrt{7}$
11). $\sqrt{80} \div \sqrt{5}$	12). $\sqrt{588} \div \sqrt{3}$	13). $\sqrt{320} \div \sqrt{10}$	14). $\sqrt{1050} \div \sqrt{3}$	15). $\sqrt{320} \div \sqrt{5}$

B. Surds in the form $a\sqrt{b}$ can be divided.

E.g. $8\sqrt{6} \div 2\sqrt{3} = 4\sqrt{2}$

Work out the following. Simplify where possible.

1). $6\sqrt{15} \div 2\sqrt{3}$	2). $14\sqrt{3} \div 7\sqrt{3}$	3). $8\sqrt{6} \div 2\sqrt{3}$	4). $20\sqrt{15} \div 4\sqrt{5}$
5). $10\sqrt{2} \div 2\sqrt{2}$	6). $15\sqrt{7} \div 3\sqrt{7}$	7). $10\sqrt{30} \div 5\sqrt{5}$	8). $18\sqrt{32} \div 3\sqrt{8}$
9). $27\sqrt{24} \div 3\sqrt{8}$	10). $24\sqrt{28} \div 3\sqrt{2}$	11). $2\sqrt{27} \div 2\sqrt{3}$	12). $21\sqrt{3} \div 3\sqrt{3}$
13). $32\sqrt{35} \div 4\sqrt{5}$	14). $10\sqrt{24} \div 2\sqrt{3}$	15). $8\sqrt{32} \div 4\sqrt{2}$	16). $4\sqrt{48} \div 4\sqrt{3}$
17). $2\sqrt{10} \div 2\sqrt{2}$	18). $12\sqrt{28} \div 3\sqrt{7}$	19). $15\sqrt{30} \div 5\sqrt{5}$	20). $30\sqrt{150} \div 5\sqrt{6}$

Mixed Questions.

A. Work out the following. Simplify where possible.

1). $(\sqrt{2})^3$ 2). $(\sqrt{3})^3$ 3). $(\sqrt{2})^5$ 4). $(\sqrt{3})^4$ 5). $(\sqrt{5})^5$

6). $(3\sqrt{2})^2$ 7). $(2\sqrt{7})^2$ 8). $(2\sqrt{3})^3$ 9). $(2\sqrt{2})^3$ 10). $(2\sqrt{3})^2$

11). $(2\sqrt{5})^2$ 12). $(5\sqrt{3})^2$ 13). $(2\sqrt{5})^3$ 14). $(3\sqrt{6})^2$ 15). $(3\sqrt{5})^3$

B. Given that $\sqrt{2} = 1.41$, $\sqrt{3} = 1.73$ and $\sqrt{5} = 2.24$ find the values of each of the following:

1). $\sqrt{18}$ 2). $\sqrt{8}$ 3). $\sqrt{48}$ 4). $\sqrt{12}$ 5). $\sqrt{75}$

6). $\sqrt{20}$ 7). $\sqrt{32}$ 8). $\sqrt{27}$ 9). $\sqrt{50}$ 10). $\sqrt{45}$

11). $\sqrt{72}$ 12). $\sqrt{98}$ 13). $\sqrt{108}$ 14). $\sqrt{80}$ 15). $\sqrt{125}$

16). True or false ? To calculate the square roots of **all** the whole numbers from 1 to 100 you only need the square roots of all the prime numbers between 1 and 100.

C. Simplify

1). $\sqrt{8} + \sqrt{2}$ 2). $\sqrt{20} - \sqrt{5}$ 3). $\sqrt{3} + \sqrt{12}$ 4). $\sqrt{8} - \sqrt{2}$

5). $\sqrt{27} + \sqrt{12}$ 6). $\sqrt{125} - \sqrt{20}$ 7). $\sqrt{48} + \sqrt{75}$ 8). $\sqrt{18} + \sqrt{72}$

9). $\sqrt{75} - \sqrt{27}$ 10). $\sqrt{80} - \sqrt{20}$ 11). $\sqrt{108} - \sqrt{27}$ 12). $\sqrt{27} - \sqrt{12}$

13). $\sqrt{147} - \sqrt{108}$ 14). $\sqrt{48} - \sqrt{27}$ 15). $\sqrt{98} + \sqrt{8} + \sqrt{2}$ 16). $\sqrt{99} - \sqrt{44} - \sqrt{11}$

17). $3\sqrt{2} - \sqrt{18}$ 18). $\sqrt{175} - 4\sqrt{7}$ 19). $3\sqrt{8} + \sqrt{50}$ 20). $5\sqrt{5} + \sqrt{20}$

21). $2\sqrt{45} + 3\sqrt{20}$ 22). $3\sqrt{32} - 2\sqrt{18}$

D. Eg. Rationalise $\frac{4}{\sqrt{2}} = \frac{4 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{4\sqrt{2}}{2} = \mathbf{2\sqrt{2}}$

Rationalise means make the denominator a rational number.

Rationalise

1). $\frac{3}{\sqrt{3}}$ 2). $\frac{10}{\sqrt{5}}$ 3). $\frac{21}{\sqrt{7}}$ 4). $\frac{8}{\sqrt{2}}$ 5). $\frac{24}{\sqrt{6}}$

6). $\frac{1}{\sqrt{3}}$ 7). $\frac{1}{\sqrt{2}}$ 8). $\frac{1}{\sqrt{5}}$ 9). $\frac{2}{\sqrt{3}}$ 10). $\frac{9}{\sqrt{15}}$

11). $\frac{21}{\sqrt{6}}$ 12). $\frac{8}{\sqrt{18}}$ 13). $\frac{2}{\sqrt{5}}$ 14). $\frac{9}{\sqrt{6}}$ 15). $\frac{30}{\sqrt{75}}$

16). $\frac{\sqrt{12}}{\sqrt{50}}$ 17). $\frac{\sqrt{12}}{\sqrt{3}}$ 18). $\frac{3\sqrt{2}}{\sqrt{10}}$ 19). $\frac{3\sqrt{7}}{\sqrt{21}}$ 20). $\frac{4\sqrt{5}}{\sqrt{20}}$

Eg. Rationalise $\frac{1}{\sqrt{7} - \sqrt{5}} = \frac{1 \times (\sqrt{7}+\sqrt{5})}{(\sqrt{7}-\sqrt{5})(\sqrt{7}+\sqrt{5})} = \frac{\sqrt{7}+\sqrt{5}}{7 - 5} = \mathbf{\frac{\sqrt{7} + \sqrt{5}}{2}}$

Eg. Rationalise $\frac{1}{\sqrt{2} + 1} = \frac{1 \times (\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)} = \frac{\sqrt{2} - 1}{2 - 1} = \mathbf{\sqrt{2} - 1}$

21). $\frac{1}{\sqrt{5} + \sqrt{2}}$ 22). $\frac{1}{\sqrt{3} - \sqrt{2}}$ 23). $\frac{4}{\sqrt{7} + \sqrt{5}}$ 24). $\frac{6}{\sqrt{13} - \sqrt{7}}$ 25). $\frac{4}{\sqrt{5} + \sqrt{3}}$

26). $\frac{7}{\sqrt{3} + 2}$ 27). $\frac{4}{\sqrt{11} - 3}$ 28). $\frac{12}{\sqrt{7} + 3}$ 29). $\frac{6}{\sqrt{13} - 2}$ 30). $\frac{6}{\sqrt{24} - \sqrt{6}}$

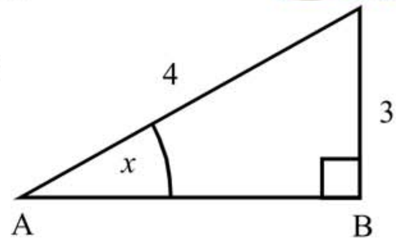
Using Surds in Trigonometry

Example

If $\sin x = \frac{3}{4}$, find $\tan x$ and $\cos x$.

This can be done without calculating the angle x. Just remember

$\sin x = \dfrac{opposite}{hypotenuse}$ So, we could draw a triangle like this:



We can now use Pythagoras' theorem to calculate AB.

$$\begin{aligned} AB^2 &= 4^2 - 3^2 \\ &= 7 \\ AB &= \sqrt{7} \end{aligned}$$

So, $\tan x = \dfrac{3}{\sqrt{7}}$ and $\cos x = \dfrac{\sqrt{7}}{4}$

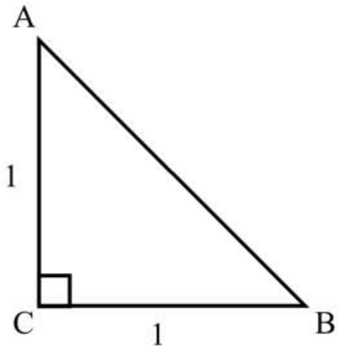
Exercise

Copy and complete the table below using a similar method to the one above.
Give all answers in surd form.

Sin x	Cos x	Tan x
$\frac{3}{5}$		
	$\frac{1}{3}$	
		$\frac{5}{2}$
		$\frac{\sqrt{7}}{2}$
$\frac{1}{\sqrt{5}}$		
	$\frac{2}{\sqrt{13}}$	
		$\frac{\sqrt{5}}{\sqrt{2}}$
	$\frac{\sqrt{7}}{3}$	
$\frac{4\sqrt{3}}{9}$		
		$\frac{\sqrt{3}+1}{8-\sqrt{5}}$

Trigonometric Ratios for 45º, 30º and 60º in Surd Form

Task 1



This is a right-angled, isosceles triangle.
Two sides have length 1 unit.

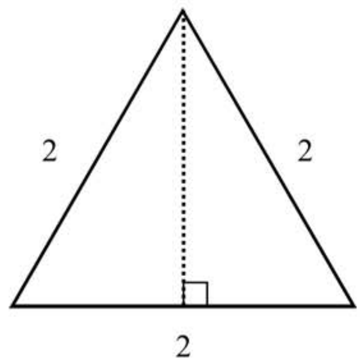
Use Pythagoras' theorem to find the length of the hypotenuse AB. Leave your answer as a surd.

Using surds where necessary, write down the ratios for sin 45º, cos 45º and tan 45º.

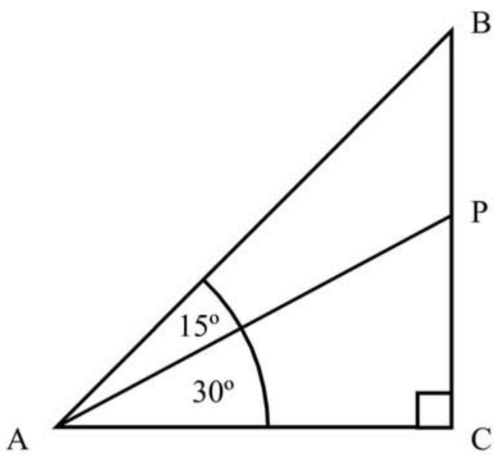
Check your answers using a calculator.

Task 2

Here is an equilateral triangle. Each side is 2 units in length. Find the height of the triangle (in surd form) using Pythagoras' theorem and use this to express the sine, cosine and tangent of 60° and 30°.



Task 3



In triangle ABC, AC = BC = 1 unit.

Use your value of tan 30º from task 2 to find the lengths of CP and BP.

Show that AP = $\dfrac{2}{\sqrt{3}}$ units.

Using the Sine rule for triangle ABP, show that

$$\sin 15^{\circ} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

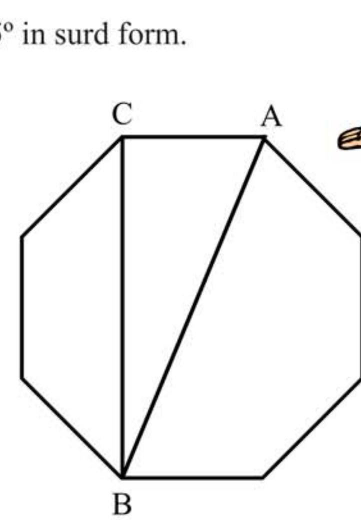
Use the value of sin 15º to find cosine and tangent of 15º in surd form.

Task 4

This is a unit octagon. Each side is 1 unit.

Show that $\angle BAC = 67\frac{1}{2}^{\circ}$ and $\angle ABC = 22\frac{1}{2}^{\circ}$

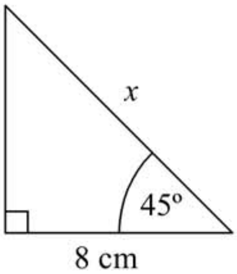
Try to find the tangent, sine and cosine of $67\frac{1}{2}^{\circ}$ and $22\frac{1}{2}^{\circ}$ in surd form.



Using Trigonometric ratios for 45°, 30° and 60° in Surd Form

Find the missing length marked x in each diagram. **Express your answers in surd form.**

1.



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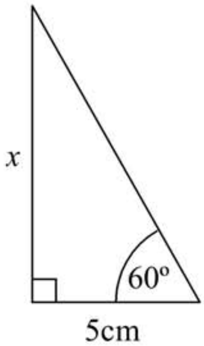

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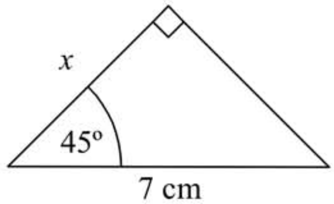

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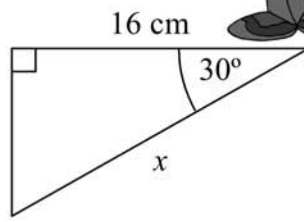

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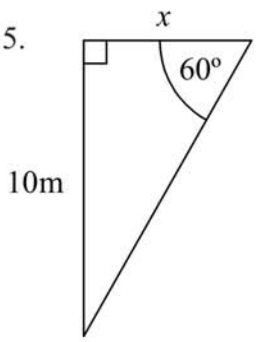

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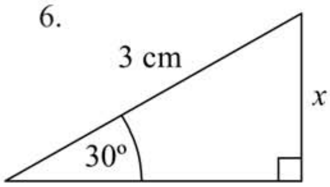

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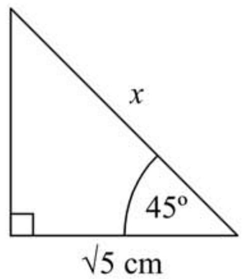

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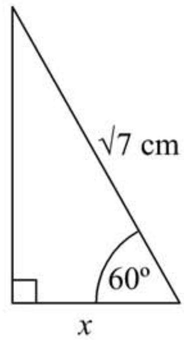

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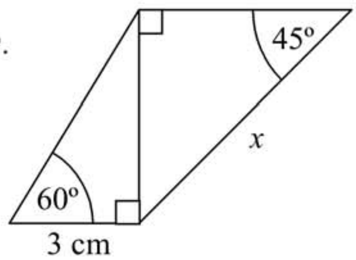

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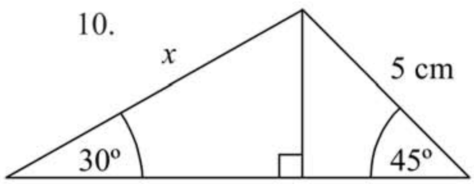

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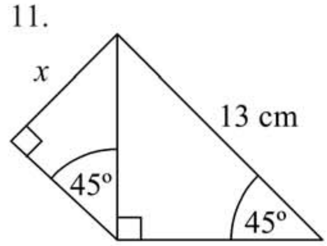

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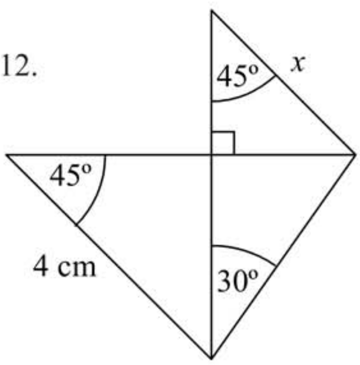

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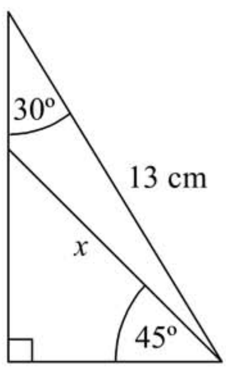

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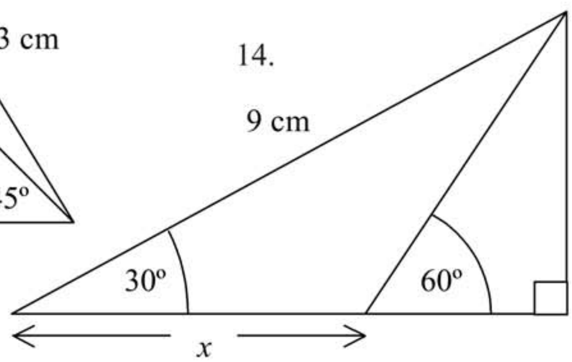

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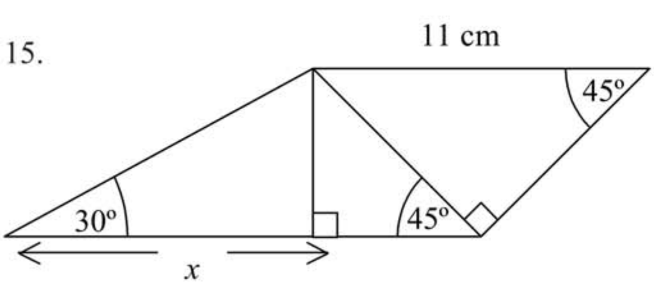

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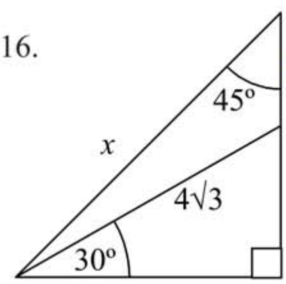

Page 27. Surds.

A.	1).	$2\sqrt{2}$	2).	$3\sqrt{3}$	3).	$2\sqrt{5}$	4).	$4\sqrt{2}$	5).	$4\sqrt{5}$
	6).	$2\sqrt{11}$	7).	$5\sqrt{3}$	8).	$6\sqrt{2}$	9).	$3\sqrt{5}$	10).	$6\sqrt{3}$
	11).	$2\sqrt{7}$	12).	$5\sqrt{5}$	13).	$7\sqrt{5}$	14).	$8\sqrt{3}$	15).	$9\sqrt{5}$
	16).	$4\sqrt{7}$	17).	$3\sqrt{7}$	18).	$6\sqrt{5}$	19).	$3\sqrt{11}$	20).	$4\sqrt{3}$
B.	1).	$\sqrt{18}$	2).	$\sqrt{20}$	3).	$\sqrt{72}$	4).	$\sqrt{80}$	5).	$\sqrt{27}$
	6).	$\sqrt{12}$	7).	$\sqrt{125}$	8).	$\sqrt{98}$	9).	$\sqrt{108}$	10).	$\sqrt{275}$
	11).	$\sqrt{243}$	12).	$\sqrt{700}$	13).	$\sqrt{320}$	14).	$\sqrt{363}$	15).	$\sqrt{1350}$
C.	1).	$3\sqrt{2}$	2).	$2\sqrt{3}$	3).	$5\sqrt{2}$	4).	$2\sqrt{10}$	5).	$2\sqrt{5}$
	6).	3	7).	4	8).	$2\sqrt{7}$	9).	$3\sqrt{2}$	10).	$5\sqrt{3}$
	11).	$2\sqrt{6}$	12).	5	13).	6	14).	6	15).	$5\sqrt{6}$
D.	1).	$8\sqrt{21}$	2).	$6\sqrt{10}$	3).	18	4).	$35\sqrt{6}$	5).	40
	6).	$12\sqrt{15}$	7).	$20\sqrt{10}$	8).	$12\sqrt{6}$	9).	24	10).	48
	11).	$30\sqrt{2}$	12).	$6\sqrt{21}$	13).	$40\sqrt{5}$	14).	$48\sqrt{2}$	15).	$60\sqrt{7}$
	16).	$24\sqrt{30}$	17).	$24\sqrt{10}$	18).	$96\sqrt{5}$	19).	$24\sqrt{10}$	20).	3

Dividing Surds.

A.	1).	$\sqrt{5}$	2).	2	3).	$5\sqrt{2}$	4).	$3\sqrt{5}$	5).	$2\sqrt{3}$

	6).	$\sqrt{6}$	7).	$\sqrt{11}$	8).	$3\sqrt{2}$	9).	$2\sqrt{2}$	10).	$3\sqrt{2}$
	11).	4	12).	14	13).	$4\sqrt{2}$	14).	$5\sqrt{14}$	15).	8
B.	1).	$3\sqrt{5}$	2).	2	3).	$4\sqrt{2}$	4).	$5\sqrt{3}$	5).	5
	6).	5	7).	$2\sqrt{6}$	8).	12	9).	$9\sqrt{3}$	10).	$8\sqrt{14}$
	11).	3	12).	7	13).	$8\sqrt{7}$	14).	$10\sqrt{2}$	15).	8
	16).	4	17).	$\sqrt{5}$	18).	8	19).	$3\sqrt{6}$	20).	30

Page 28.

A.	1).	$2\sqrt{2}$	2).	$3\sqrt{3}$	3).	$4\sqrt{2}$	4).	9	5).	$25\sqrt{5}$
	6).	18	7).	28	8).	$24\sqrt{3}$	9).	$16\sqrt{2}$	10).	12
	11).	20	12).	75	13).	$40\sqrt{5}$	14).	54	15).	$135\sqrt{5}$
B.	1).	4.23	2).	2.82	3).	6.92	4).	3.46	5).	8.65
	6).	4.48	7).	5.64	8).	5.19	9).	7.05	10).	6.72
	11).	8.46	12).	9.87	13).	10.38	14).	8.96	15).	11.2
	16).	True								
C.	1).	$3\sqrt{2}$	2).	$\sqrt{5}$	3).	$3\sqrt{3}$	4).	$\sqrt{2}$	5).	$5\sqrt{3}$
	6).	$3\sqrt{5}$	7).	$9\sqrt{3}$	8).	$9\sqrt{2}$	9).	$2\sqrt{3}$	10).	$2\sqrt{5}$
	11).	$3\sqrt{3}$	12).	$\sqrt{3}$	13).	$\sqrt{3}$	14).	$\sqrt{3}$	15).	$10\sqrt{2}$
	16).	0	17).	0	18).	$\sqrt{7}$	19).	$11\sqrt{2}$	20).	$7\sqrt{5}$
	21).	$12\sqrt{5}$	22).	$6\sqrt{2}$						
D.	1).	$\sqrt{3}$	2).	$2\sqrt{5}$	3).	$3\sqrt{7}$	4).	$4\sqrt{2}$	5).	$4\sqrt{6}$
	6).	$\sqrt{3}/3$	7).	$\sqrt{2}/2$	8).	$\sqrt{5}/5$	9).	$2\sqrt{3}/3$	10).	$3\sqrt{15}/5$
	11).	$7\sqrt{6}/2$	12).	$4\sqrt{2}/3$	13).	$2\sqrt{5}/5$	14).	$3\sqrt{6}/2$	15).	$2\sqrt{3}$
	16).	$\sqrt{6}/5$	17).	2	18).	$3\sqrt{5}/5$	19).	$\sqrt{3}$	20).	2
	21).	$(\sqrt{5} - \sqrt{2})/3$	22).	$\sqrt{3} + \sqrt{2}$	23).	$2\sqrt{7} - 2\sqrt{5}$	24).	$\sqrt{13}+ \sqrt{7}$	25).	$2\sqrt{5}-2\sqrt{3}$
	26).	$\sqrt{3}-2$	27).	$2\sqrt{11} + 6$	28).	$-6\sqrt{7} + 18$	29).	$(2\sqrt{13}+ 4)/3$	30).	$\sqrt{6}$

Page 33. Using Surds in Trigonometry.

Sin x	Cos x	Tan x
—	$\frac{4}{5}$	$\frac{3}{4}$
$\frac{2\sqrt{2}}{3}$	—	$2\sqrt{2}$
$\frac{5}{\sqrt{29}}$	$\frac{2}{\sqrt{29}}$	—
$\frac{\sqrt{7}}{\sqrt{11}}$	$\frac{2}{\sqrt{11}}$	—
—	$\frac{2}{\sqrt{5}}$	$\frac{1}{2}$
$\frac{3}{\sqrt{13}}$	—	$\frac{3}{2}$
$\frac{\sqrt{5}}{\sqrt{7}}$	$\frac{\sqrt{2}}{\sqrt{7}}$	—
$\frac{\sqrt{2}}{3}$	—	$\frac{\sqrt{2}}{\sqrt{7}}$
—	$\frac{\sqrt{33}}{9}$	$\frac{4\sqrt{3}}{\sqrt{33}}$
$\frac{\sqrt{3}+1}{\sqrt{(73+2\sqrt{3}-16\sqrt{5})}}$	$\frac{8-\sqrt{5}}{\sqrt{(73+2\sqrt{3}-16\sqrt{5})}}$	—

Page 34. Trigonometric Ratios for 45°, 30° and 60° in Surd Form

Task 1 AB = $\sqrt{2}$ Hence,

Sin 45° = $1/\sqrt{2}$, Cos 45° = $1/\sqrt{2}$ and Tan 45° = 1.

Task 2 Height of triangle = $\sqrt{3}$ Hence,

Sin 30° = 1/2, Cos 30° = $\sqrt{3}/2$ and Tan 30° = $1/\sqrt{3}$.

Sin 60° = $\sqrt{3}/2$, Cos 60° = 1/2 and Tan 60° = $\sqrt{3}$.

Task 3 Tan 30° = CP/AC = 1/√3 and AC = 1,
so CP = 1/√3 and BP = 1 - (1/√3)
Use Pythagoras to find AP = 2/√3

Sine rule for triangle ABP: $\dfrac{\sin 15^\circ}{1-\frac{1}{\sqrt{3}}} = \dfrac{\sin 45^\circ}{\frac{2}{\sqrt{3}}}$

Use your value for sin 45° from task 1 and rearrange to sin 15°.

$\cos 15^\circ = \dfrac{\sqrt{4+2\sqrt{3}}}{2\sqrt{2}}$ and $\tan 15^\circ = \dfrac{\sqrt{3}-1}{\sqrt{4+2\sqrt{3}}}$

Task 4

Angle BAC is 67½° beacuse it is half of the interior angle of the octagon (135°).
Find ABC using angles in a triangle.

The key to the next part is to find the length BC = $1+\dfrac{1}{\sqrt{2}}$. Join horizontally opposite corners of the octagon and use the right-angled triangles produced on BC.

Pythagoras gives AB = $\dfrac{3+2\sqrt{2}}{2}$. Hence,

Sin 67½º = $\dfrac{2+\sqrt{2}}{3+2\sqrt{2}}$ Cos 67½º = $\dfrac{1}{3+2\sqrt{2}}$ Tan 67½º = $1+\dfrac{1}{\sqrt{2}}$

Sin22½º = $\dfrac{1}{3+2\sqrt{2}}$ Sin22½º = $\dfrac{2+\sqrt{2}}{3+2\sqrt{2}}$ Tan 67½º = $\dfrac{2}{2+\sqrt{2}}$

Page 35. Using Trigonometric Ratios for 45º, 30º and 60º in Surd Form

1. $x = 8\sqrt{2}$ 2. $x = 5\sqrt{3}$ 3. $x = \dfrac{7}{\sqrt{2}}$ 4. $x = \dfrac{32}{\sqrt{3}}$

5. $x = \dfrac{10}{\sqrt{3}}$ 6. $x = 3/2$ 7. $x = \sqrt{10}$ 8. $x = \dfrac{\sqrt{7}}{2}$ 9. $x = 3\sqrt{6}$

10. $x = 5\sqrt{2}$ 11. $x = 6.5$ 12. $x = \dfrac{4}{\sqrt{3}}$ 13. $x = \dfrac{13}{\sqrt{2}}$ 14. $x = 3\sqrt{3}$

15. $x = \dfrac{11\sqrt{3}}{2}$ 16. $x = 6\sqrt{2}$

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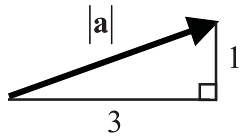
Summer 2020 Further Maths Transition work

Worksheet B – Vectors (and answers)

Magnitude of a Vector

The magnitude, or modulus, of a vector is just a term for its length. The length of a column vector can be found using Pythagoras' theorem. For example, consider the vector $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

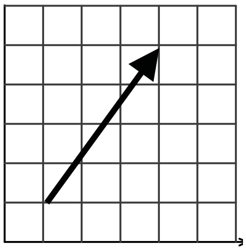
The notation for the length of **a** is $|\mathbf{a}|$



Using Pythagoras' theorem $|\mathbf{a}|^2 = 3^2 + 1^2 \Rightarrow\ = \sqrt{10}$.

A. Find the magnitudes of the following vectors, leaving your answer in surd form if appropriate.

1.



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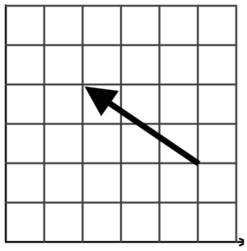

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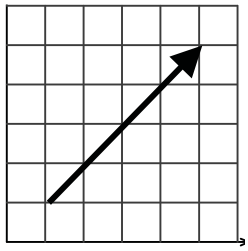

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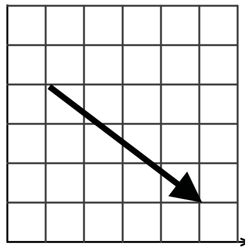

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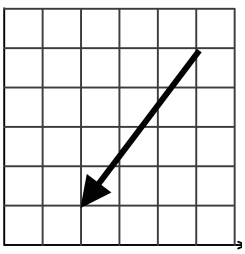

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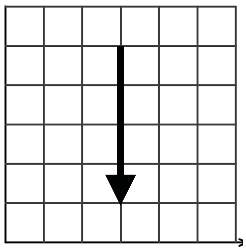

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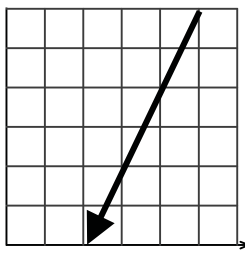

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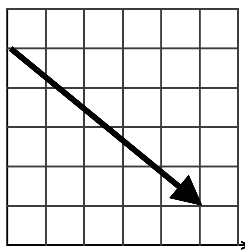

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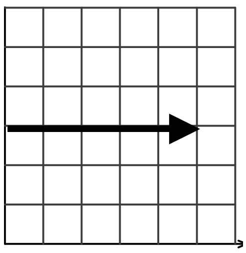

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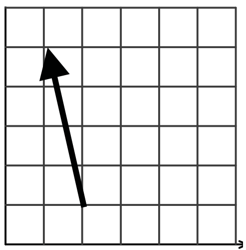

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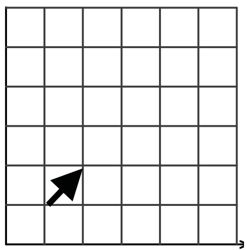

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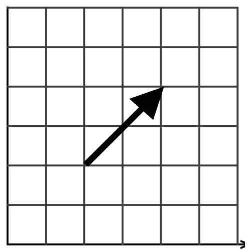

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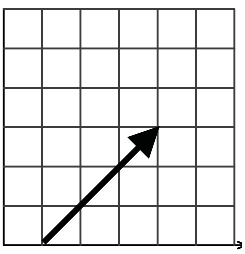

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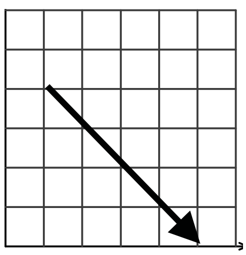

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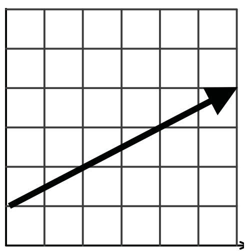

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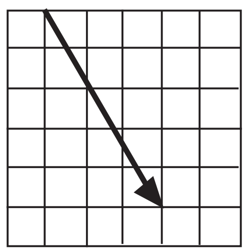


It is useful if you can calculate the length of a vector without drawing a diagram. Pythagoras' theorem gives the following formula.

$$\left|\begin{pmatrix}a\\b\end{pmatrix}\right| = \sqrt{a^2+b^2}\,. \qquad \textbf{E.g.}\ \left|\begin{pmatrix}4\\-2\end{pmatrix}\right| = \sqrt{4^2+(-2)^2} = \sqrt{20} \quad = 2\sqrt{5}$$

B. Find the lengths of the following vectors, leaving your answer in surd form if appropriate.

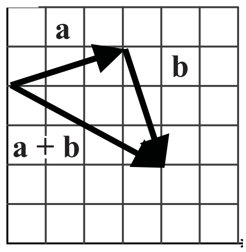
1. $\begin{pmatrix}3\\4\end{pmatrix}$ **2.** $\begin{pmatrix}-4\\3\end{pmatrix}$ **3.** $\begin{pmatrix}5\\12\end{pmatrix}$ **4.** $\begin{pmatrix}-2\\3\end{pmatrix}$ **5.** $\begin{pmatrix}-2\\4\end{pmatrix}$ **6.** $\begin{pmatrix}1\\1\end{pmatrix}$ **7.** $\begin{pmatrix}1\\-1\end{pmatrix}$

8. $\begin{pmatrix}2\\1\end{pmatrix}$ **9.** $\begin{pmatrix}-1\\2\end{pmatrix}$ **10.** $\begin{pmatrix}7\\24\end{pmatrix}$ **11.** $\begin{pmatrix}-7\\-1\end{pmatrix}$ **12.** $\begin{pmatrix}-1\\-2\end{pmatrix}$ **13.** $\begin{pmatrix}5\\-5\end{pmatrix}$ **14.** $\begin{pmatrix}8\\-4\end{pmatrix}$

C. The diagram shows three vectors, **a**, **b** and **a** + **b**. Find the length of each vector and use Pythagoras's theorem to show that **a** is perpendicular to **b**.

Solution:

$\mathbf{a} = \begin{pmatrix}3\\1\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}1\\-3\end{pmatrix}$ and $\mathbf{a}+\mathbf{b} = \begin{pmatrix}4\\-2\end{pmatrix}$.



$|\mathbf{a}| = \sqrt{10},\qquad |\mathbf{b}| = \sqrt{10},\qquad |\mathbf{a}+\mathbf{b}| = \sqrt{20}\,.$

Using Pythagoras' theorem $\left(\sqrt{10}\right)^2 + \left(\sqrt{10}\right)^2 = 10+10 = 20 = \left(\sqrt{20}\right)^2$

therefore **a** and **b** are perpendicular.

1. In each of the following cases draw a diagram showing **a**, **b** and **a** + **b**, find the length of each vector, and use Pythagoras' theorem to verify that **a** is perpendicular to **b**.

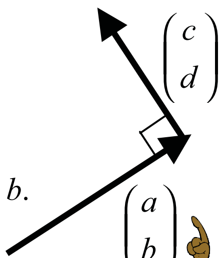
(i) $\mathbf{a} = \begin{pmatrix}1\\1\end{pmatrix}, \mathbf{b} = \begin{pmatrix}1\\-1\end{pmatrix}$ **(ii)** $\mathbf{a} = \begin{pmatrix}2\\1\end{pmatrix}, \mathbf{b} = \begin{pmatrix}-1\\2\end{pmatrix}$ **(iii)** $\mathbf{a} = \begin{pmatrix}3\\-2\end{pmatrix}, \mathbf{b} = \begin{pmatrix}-2\\-3\end{pmatrix}$

(iv) $\mathbf{a} = \begin{pmatrix}-2\\4\end{pmatrix}, \mathbf{b} = \begin{pmatrix}2\\1\end{pmatrix}$ **(v)** $\mathbf{a} = \begin{pmatrix}1\\4\end{pmatrix}, \mathbf{b} = \begin{pmatrix}4\\-1\end{pmatrix}$ **(vi)** $\mathbf{a} = \begin{pmatrix}6\\3\end{pmatrix}, \mathbf{b} = \begin{pmatrix}-1\\2\end{pmatrix}$

(vii) $\mathbf{a} = \begin{pmatrix}2\\-2\end{pmatrix}, \mathbf{b} = \begin{pmatrix}-2\\-2\end{pmatrix}$ **(viii)** $\mathbf{a} = \begin{pmatrix}8\\-2\end{pmatrix}, \mathbf{b} = \begin{pmatrix}-2\\-8\end{pmatrix}$ **(ix)** $\mathbf{a} = \begin{pmatrix}8\\4\end{pmatrix}, \mathbf{b} = \begin{pmatrix}-1\\2\end{pmatrix}$

2. Use Pythagoras' theorem to prove that the vectors $\begin{pmatrix}x\\y\end{pmatrix}$ and $\begin{pmatrix}-y\\x\end{pmatrix}$ are perpendicular.

3. The diagram shows the two vectors $\begin{pmatrix}a\\b\end{pmatrix}$ and $\begin{pmatrix}c\\d\end{pmatrix}$.



(a) Write down the sum of these vectors as a column vector.

(b) Write down, in surd form, the length of $\begin{pmatrix}a\\b\end{pmatrix}$, in terms of a and b.

(c) Write down, in surd form, the length of $\begin{pmatrix}c\\d\end{pmatrix}$, in terms of c and d.

(d) Write down, in surd form, the length of the sum of these vectors in terms of a, b, c, and d.

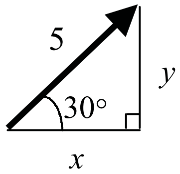
(e) If the vectors are perpendicular prove that $ac + bd = 0$.

Components

Sometimes we know the length and direction of a vector but need to know its column vector (component) form.

Find the following vectors as column vectors.
Give answers to one decimal place.

The vector of length 5 at 30° to the x-axis can be split into two components using trigonometry.

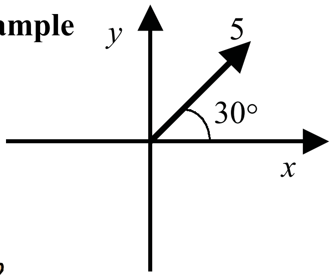


$$\cos 30° = \frac{x}{5} \Rightarrow x = 5\cos 30 = 4.3$$

$$\sin 30° = \frac{y}{5} \Rightarrow y = 5\sin 30 = 2.5$$

So the vector is $\begin{pmatrix} 4.3 \\ 2.5 \end{pmatrix}$

Example



Diagrams not to scale

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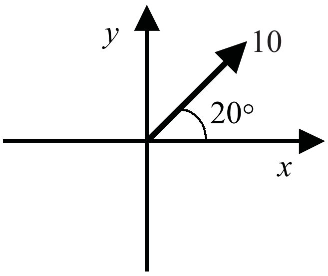

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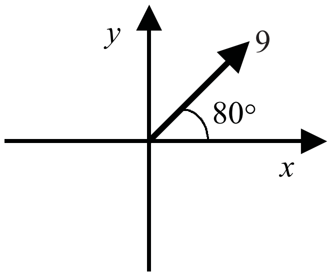

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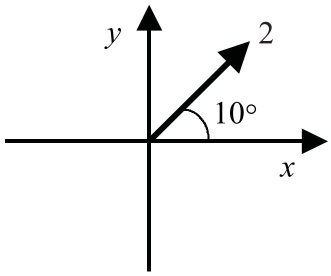

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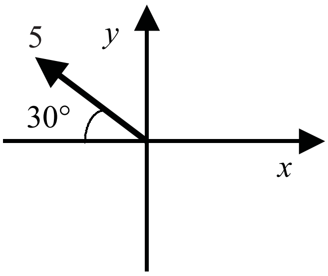

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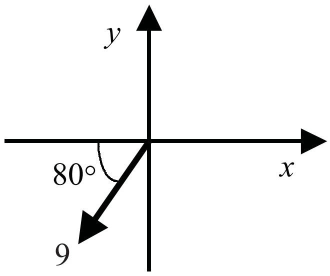

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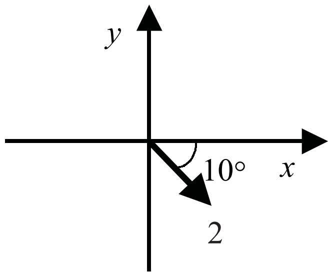

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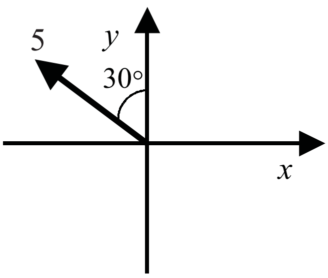

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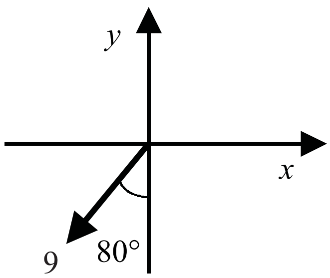

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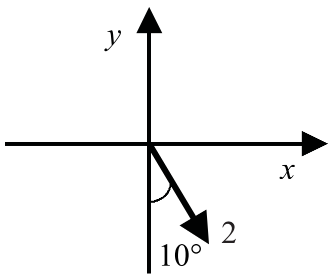

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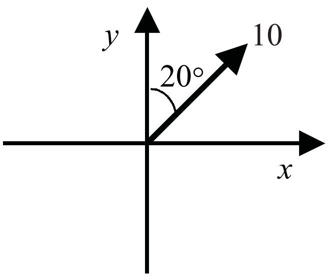

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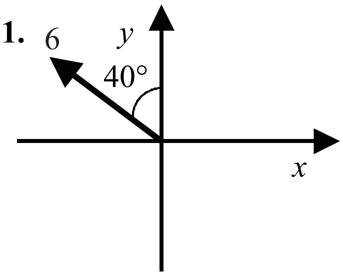

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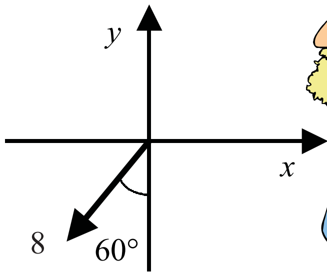

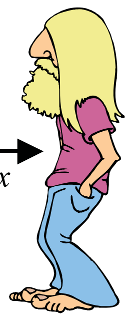

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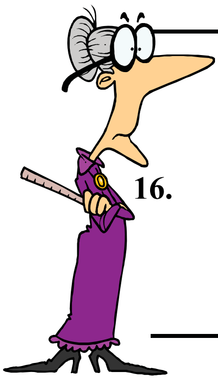

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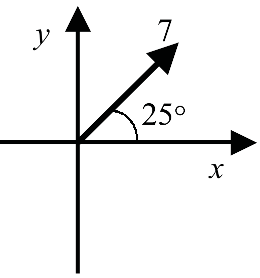

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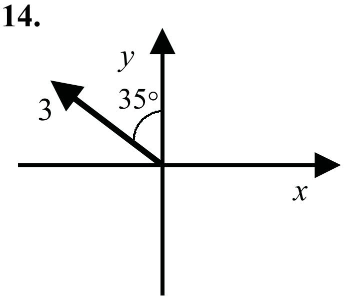

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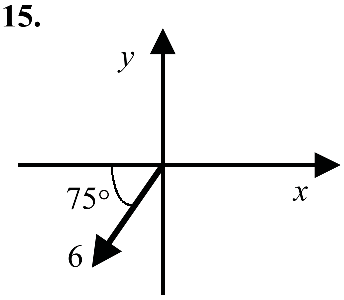

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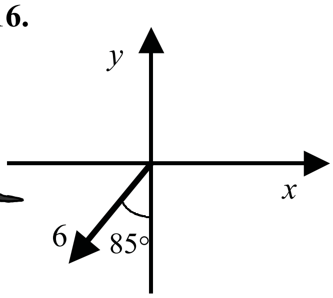

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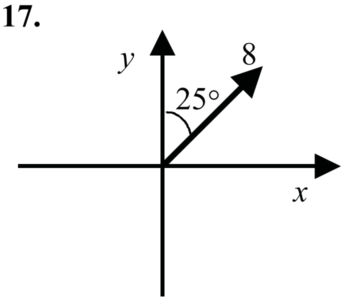

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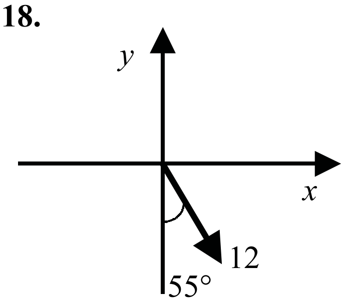

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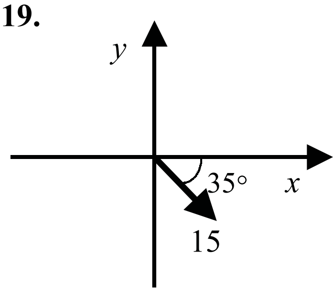

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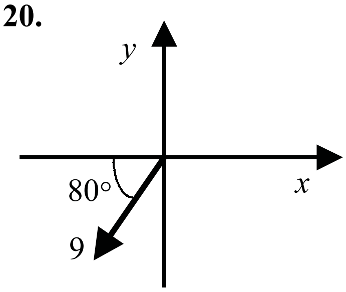

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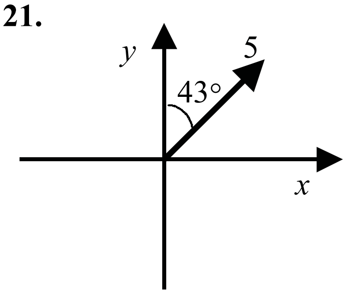

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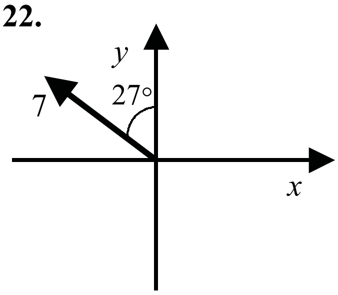

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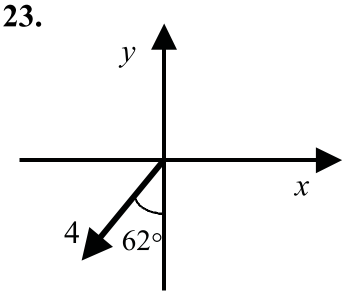

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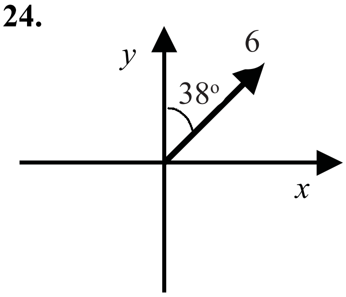
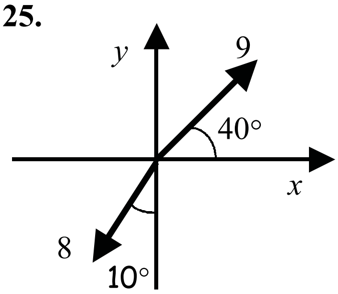


In the last six questions find each vector in column form and hence add the vectors together.

25.



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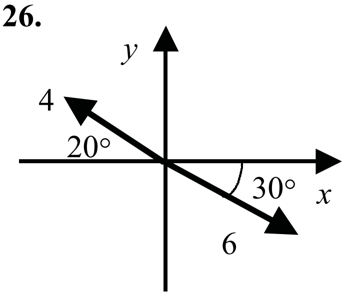

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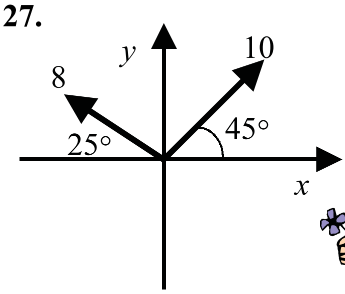

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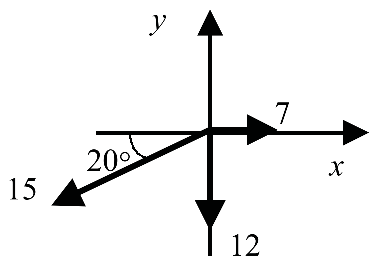

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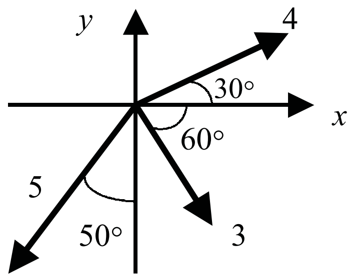

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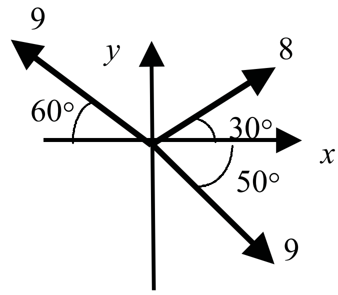

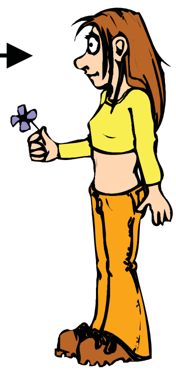

Page 8. Magnitude of a Vector

A. **1.** 5 **2.** $\sqrt{13}$ **3.** $4\sqrt{2}$ **4.** 5 **5.** 5 **6.** 4 **7.** $3\sqrt{5}$

8. $\sqrt{41}$ **9.** 5 **10.** $\sqrt{17}$ **11.** $\sqrt{2}$ **12.** $2\sqrt{2}$ **13.** $3\sqrt{2}$ **14.** $4\sqrt{2}$

15. $3\sqrt{5}$ **16.** $\sqrt{34}$

Page 9.

B. **1.** 5 **2.** 5 **3.** 13 **4.** $\sqrt{13}$ **5.** $2\sqrt{5}$ **6.** $\sqrt{2}$ **7.** $\sqrt{2}$

8. $\sqrt{5}$ **9.** $\sqrt{5}$ **10.** 25 **11.** $5\sqrt{2}$ **12.** $\sqrt{5}$ **13.** $5\sqrt{2}$ **14.** $4\sqrt{5}$

C. **1.** **(i)** $\sqrt{2}, \sqrt{2}, 2$ **(ii)** $\sqrt{5}, \sqrt{5}, \sqrt{10}$ **(iii)** $\sqrt{13}, \sqrt{13}, \sqrt{26}$ **(iv)** $\sqrt{20}, \sqrt{5}, 5$

(v) $\sqrt{17}, \sqrt{17}, \sqrt{34}$ **(vi)** $3\sqrt{5}, \sqrt{5}, 5\sqrt{2}$ **(vii)** $2\sqrt{2}, 2\sqrt{2}, 4$ **(viii)** $2\sqrt{17}, 2\sqrt{17}, 2\sqrt{34}$

(ix) $4\sqrt{5}, \sqrt{5}, \sqrt{85}$

3. (a) $\begin{pmatrix} a+c \\ b+d \end{pmatrix}$ **(b)** $\sqrt{a^2+b^2}$ **(c)** $\sqrt{c^2+d^2}$ **(d)** $\sqrt{(a+c)^2+(b+d)^2}$

Page 11. Components

1. $\begin{pmatrix} 9.4 \\ 3.4 \end{pmatrix}$ **2.** $\begin{pmatrix} 1.6 \\ 8.9 \end{pmatrix}$ **3.** $\begin{pmatrix} 2.0 \\ 0.3 \end{pmatrix}$ **4.** $\begin{pmatrix} -4.3 \\ 2.5 \end{pmatrix}$ **5.** $\begin{pmatrix} -1.6 \\ -8.9 \end{pmatrix}$ **6.** $\begin{pmatrix} 2.0 \\ -0.3 \end{pmatrix}$ **7.** $\begin{pmatrix} -2.5 \\ 4.3 \end{pmatrix}$ **8.** $\begin{pmatrix} -8.9 \\ -1.6 \end{pmatrix}$

9. $\begin{pmatrix} 0.3 \\ -2.0 \end{pmatrix}$ **10.** $\begin{pmatrix} 3.4 \\ 9.4 \end{pmatrix}$ **11.** $\begin{pmatrix} -3.9 \\ 4.6 \end{pmatrix}$ **12.** $\begin{pmatrix} -6.9 \\ -4 \end{pmatrix}$

Page 12.

13. $\begin{pmatrix} 6.3 \\ 3.0 \end{pmatrix}$ **14.** $\begin{pmatrix} -1.7 \\ 2.5 \end{pmatrix}$ **15.** $\begin{pmatrix} -1.6 \\ -5.8 \end{pmatrix}$ **16.** $\begin{pmatrix} -6.0 \\ -0.5 \end{pmatrix}$ **17.** $\begin{pmatrix} 3.4 \\ 7.3 \end{pmatrix}$ **18.** $\begin{pmatrix} 9.8 \\ -6.9 \end{pmatrix}$ **19.** $\begin{pmatrix} 12.3 \\ -8.6 \end{pmatrix}$ **20.** $\begin{pmatrix} -1.6 \\ -8.9 \end{pmatrix}$

21. $\begin{pmatrix} 3.4 \\ 3.7 \end{pmatrix}$ **22.** $\begin{pmatrix} -3.2 \\ 6.2 \end{pmatrix}$ **23.** $\begin{pmatrix} -3.5 \\ -1.9 \end{pmatrix}$ **24.** $\begin{pmatrix} 3.7 \\ 4.7 \end{pmatrix}$ **25.** $\begin{pmatrix} 5.5 \\ -2.1 \end{pmatrix}$ **26.** $\begin{pmatrix} 1.4 \\ -1.6 \end{pmatrix}$ **27.** $\begin{pmatrix} -0.2 \\ 10.5 \end{pmatrix}$ **28.** $\begin{pmatrix} -7.1 \\ -17.1 \end{pmatrix}$

29. $\begin{pmatrix} 1.2 \\ -3.8 \end{pmatrix}$ **30.** $\begin{pmatrix} 8.2 \\ 4.9 \end{pmatrix}$

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Summer 2020 Further Maths Transition work

Worksheet C – Calculus (and answers)

Gradient of Curves 2

1. The gradient function on a curve has a special notation : $\frac{dy}{dx}$.

It can be shown that for the curve $y = ax^n$, $= anx^{n-1}$.

Find $\frac{dy}{dx}$ for the following curves.

(a) $y = 2x^2$ **(b)** $y = 4x^3$ **(c)** $y = 3x^5$ **(d)** $y = 12x^{10}$ **(e)** $y = -4x^2$

(f) $y = -3x^{-1}$ **(g)** $y = 3x$ **(h)** $y = 4x$ **(i)** $y = -5x$ **(j)** $y = -4x^2$

(k) $y = -3x^{-2}$ **(l)** $y = 12x - 1$ **(m)** $y = 3x + 2$ **(n)** $y = x^2 + x$ **(o)** $y = 2x^2 + 4x^3$

(p) $y = 2x^2 + 3x + 4$ **(q)** $y = 5x^2 - 6x - 2$ **(r)** $y = 1 - x^2$

(s) $y = 2x - x^3 + 4$ **(t)** $y = (x-1)(x+1)$ **(u)** $y = x(x+3)(x-1)$

(v) $y = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}$ **(w)** $y = x - 3x^{-2}$ **(x)** $y = \frac{3}{x^2} - \frac{2}{x} + 3$

(y) $y = 4x^2 + (2x-1)^2$ **(z)** $y = \left(x - \frac{1}{x}\right)^2$.

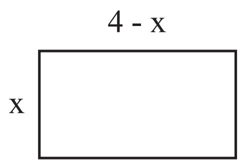
2. Find $\frac{dy}{dx}$ for the following, and by putting $\frac{dy}{dx} = 0$,
find the coordinates of the points on the curves where the gradient is zero.

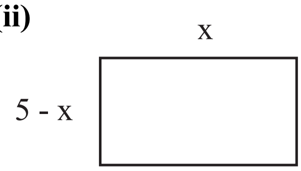
(a) $y = x^2 - 2x + 4$ **(b)** $y = x^2 + 4x + 4$ **(c)** $y = 3x^2 - 6x$

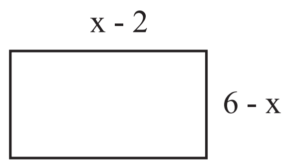
(d) $y = 3 - x^2$ **(e)** $y = 2x - x^2 - 10$ **(f)** $y = x^3 - 3x$

(g) $y = 2x^3 - 3x^2 - 12x + 2$ **(h)** $y = 9x - 3x^2 - x^3$ **(i)** $y = x^3 - x$

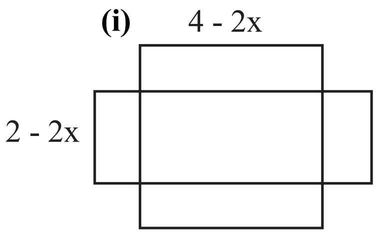
3. For each of the rectangles below
(a) find the area, A, in terms of x,
(b) find $\frac{dA}{dx}$,
(c) find the value of x which makes the area of the rectangle greatest.

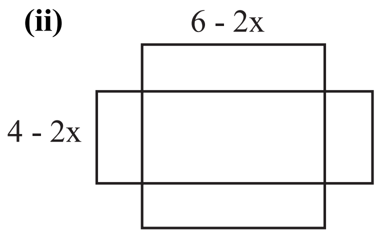
(i)


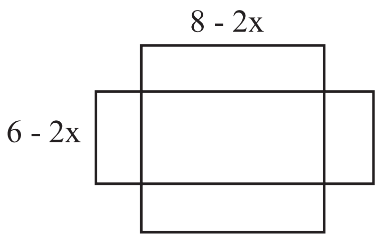
(ii)


(iii)


4. Each of the following diagrams shows a rectangle where a square (x by x) has been removed at the corners. The resulting shape can be folded to make an open box. For each one
(a) find the volume of the box so formed, V, in terms of x,
(b) find $\frac{dV}{dx}$ and the value of x which makes the volume of the box greatest.

(i)


(ii)


(iii)


Integration 1

Indefinite Integrals

Integration is the opposite of differentiation. There is a simple rule for the integration of a polynomial:

$$\int ax^n \, \mathrm{d}x = \frac{ax^{n+1}}{n+1} + c \qquad n \neq -1$$

A. Use this rule to integrate the following,

1. $3x^2$	**2.** $4x^7$	**3.** $6x^{11}$	**4.** $4x^3$	**5.** $1.5x^{0.5}$
6. $x^2 + 4x + 3$	**7.** $3x^2 + 2x + 1$	**8.** $5x^2 - 3x - 2$	**9.** $7x^3 + x^4 - 3$	**10.** $16x^{-2} - x$
11. $3x^4 - x^{16}$	**12.** $2x^5 - x^{-6}$	**13.** $x^{10} + x^{20}$	**14.** $x^2 - 1$	**15.** $x^{-0.5}$
16. $x^3 + 4x^2 + 3x$	**17.** $x^4 + 2x^2$	**18.** $x(x - 1)$	**19.** $(x - 1)(x + 1)$	**20.** $(2x - 1)(3x + 1)$

Integration can be used to find the equation of a curve when the gradient function (i.e. $\frac{\mathrm{d}y}{\mathrm{d}x}$) is known.

Example

Suppose we know that, $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 1$, and that the curve passes through the point (1,5).

We can integrate the expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ to get y.

If $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 1$, then $y = \int 3x^2 + 1 \, \mathrm{d}x \Rightarrow y = x^3 + x + c$.

> Notice the arbitrary constant, c, in this equation.
> This must be included because when $x^3 + x + c$ is differentiated, it will always give $3x^2 + 1$ for all constants, c.

We now substitute (1,5) into this expression to find c.

$5 = 1^3 + 1 + c$ therefore $c = 3$.

So the equation of the curve is $y = x^3 + x + 3$.

B. Each of the following examples gives the gradient of a curve, and a point lying on it. Find the equation of the curve in each case.

1.	$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 2x + 1$	$(1,6)$	**2.**	$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x + 3$	$(-1,1)$
3.	$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 + x^2$	$(0,2)$	**4.**	$\frac{\mathrm{d}y}{\mathrm{d}x} = x(x-1)$	$(-1,\frac{1}{6})$
5.	$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - \frac{1}{x^2}$	$(1,\frac{1}{2})$	**6.**	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}x^{\frac{1}{2}}$	$(9,1)$
7.	$\frac{\mathrm{d}y}{\mathrm{d}x} = 4 - 2x$	$(2,4)$	**8.**	$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 10x + 4$	$(1,0)$
9.	$\frac{\mathrm{d}y}{\mathrm{d}x} = -4x^3$	$(-3,0)$	**10.**	$\frac{\mathrm{d}y}{\mathrm{d}x} = 5x^4 - 3x^2$	$(0,0)$
11.	$\frac{\mathrm{d}y}{\mathrm{d}x} = 30x^2 - 10$	$(1,0)$	**12.**	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{\sqrt{x}}$	$(1,3)$

Integration 2

Definite Integrals

A definite integral has limits. The indefinite integral is found first of all, then the limits are substituted into the new expression, and finally the difference is found. The following example shows how.

$$\int_1^2 (2x^2+4x+5)\ \mathrm{d}x = \left[\tfrac{2}{3}x^3+2x^2+5x\right]_1^2$$

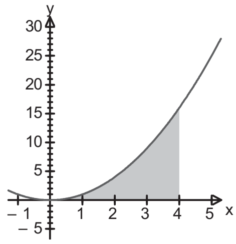
$$= \left(\tfrac{2}{3}\times 2^3+2\times 2^2+5\times 2\right)-\left(\tfrac{2}{3}\times 1^3+2\times 1^2+5\times 1\right)=15\tfrac{2}{3}$$

A. Find the following definite integrals.

1. $\int_1^2 x^2\ \mathrm{d}x$ **2.** $\int_2^3 3x^2+1\ \mathrm{d}x$ **3.** $\int_0^1 x^3\ \mathrm{d}x$ **4.** $\int_{-1}^2 5x\ \mathrm{d}x$ **5.** $\int_{-1}^1 x^2-1\ \mathrm{d}x$

6. $\int_0^2 2-x\ \mathrm{d}x$ **7.** $\int_0^1 x^2-x\ \mathrm{d}x$ **8.** $\int_{-2}^2 x\ \mathrm{d}x$ **9.** $\int_0^3 4x^3-2x\ \mathrm{d}x$ **10.** $\int_{-1}^0 x(x+1)\ \mathrm{d}x$

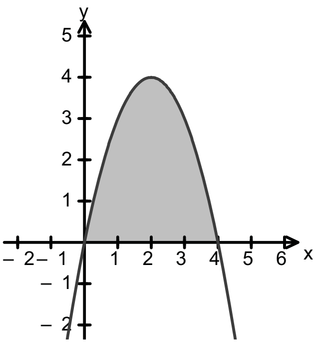
Definite integrals can be used to find areas between a curve and the x - axis. For example the shaded area in the diagram shown opposite, between the curve $y = x^2$ and the x - axis, between $x = 1$ and $x = 4$, is given by,



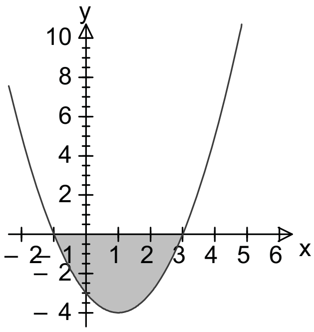
$$\int_1^4 x^2\ \mathrm{d}x = \left[\tfrac{1}{3}x^3\right]_1^4 = \tfrac{64}{3}-\tfrac{1}{3}=21$$

B. Find the areas shaded in the following diagrams

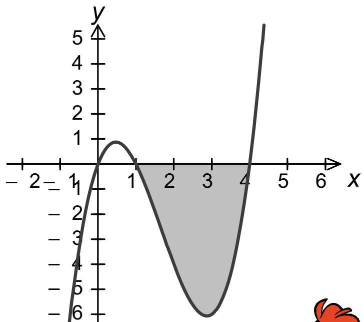
1. $y = 4x - x^2$



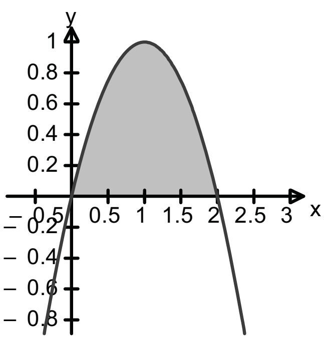
2. $y = x^2-2x-3$



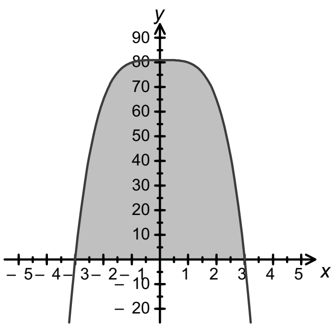
3. $y = x^3-5x^2+4x$



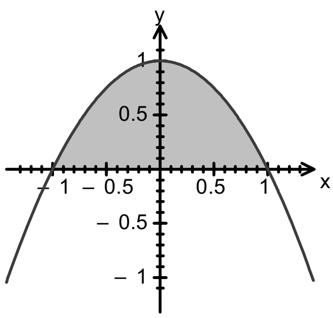
4. $y = x(2 - x)$



5. $y = 81-x^4$



6. $y = 1-x^2$



Page 41. Gradients of a Curve 2.

1. **(a)** $4x$ **(b)** $12x^2$ **(c)** $15x^4$ **(d)** $120x^9$ **(e)** $-8x$ **(f)** $3x^{-2}$ **(g)** 3

(h) 4 **(i)** -5 **(j)** $-8x$ **(k)** $6x^{-3}$ **(l)** 12 **(m)** 3 **(n)** $2x+1$

(o) $4x+12x^2$ **(p)** $4x+3$ **(q)** $10x-6$ **(r)** $-2x$ **(s)** $2-3x^2$

(t) $2x$ **(u)** $3x^2+4x-3$

(v) $-\frac{1}{x^2}-\frac{2}{x^3}-\frac{3}{x^4}$ **(w)** $1+\frac{6}{x^3}$ **(x)** $\frac{2}{x^2}-\frac{6}{x^3}$ **(y)** $16x-4$ **(z)** $2x + 2/x^3$

2. **(a)** $2x-2, x=1$, $y = 3$ **(b)** $2x+4, x=-2$, $y = 0$

(c) $6x-6, x=1$, y = -3 **(d)** $-2x, x=0$, $y = 3$

(e) $2-2x, x=1$, y = -9 **(f)** $3x^2-3, x=\pm 1$, y = 0 or -2

(g) $6x^2-6x-12, x=-1$ or 2, y = 9 or -18

(h) $9-6x-3x^2, x=-3$ or 1, $y = 5$ or -27 **(i)** $3x^2-1, x=\pm\frac{1}{\sqrt{3}}$, $y = 0$ or 0

3. a). i). $4x - x^2$ ii). $5x - x^2$ iii). $8x - 12 - x^2$

b). i). $4 - 2x$ ii). $5 - 2x$ iii). $8 - 2x$

c). i). 2 ii). 2.5 iii). 4

4. a). i). $4x^3 - 12x^2 + 8x$ ii). $4x^3 - 20x^2 + 24x$

iii). $4x^3 - 28x^2 + 48x$

b). i). $12x^2 - 24x + 8$, $x = 0.423$ ii). $12x^2 - 40x + 24$, $x = 0.785$

iii). $12x^2 - 56x + 48$, $x = 1.31$

Page 42. Integration 1.

A. **1.** x^3+c **2.** $\frac{1}{2}x^8+c$ **3.** $\frac{1}{2}x^{12}+c$

4. x^4+c **5.** $x^{1.5}+c$ **6.** $\frac{1}{3}x^3+2x^2+3x+c$

7. x^3+x^2+x+c **8.** $\frac{5}{3}x^3-\frac{3}{2}x^2-2x+c$ **9.** $\frac{7}{4}x^4+\frac{1}{5}x^5-3x+c$

10. $-\frac{16}{x}-\frac{1}{2}x^2+c$ **11.** $\frac{3}{5}x^5-\frac{1}{17}x^{17}+c$ **12.** $\frac{1}{3}x^6+\frac{1}{5}x^{-5}+c$

13. $\frac{1}{11}x^{11}+\frac{1}{20}x^{20}+c$ **14.** $\frac{1}{3}x^3-x+c$ **15.** $2x^{0.5}+c$

16. $\frac{1}{4}x^4+\frac{4}{3}x^3+\frac{3}{2}x^2+c$ **17.** $\frac{1}{5}x^5+\frac{2}{3}x^3+c$ **18.** $\frac{1}{3}x^3-\frac{1}{2}x^2+c$

19. $\frac{1}{3}x^3-x+c$ **20.** $2x^3-\frac{1}{2}x^2-x+c$

B. **1.** $y=x^3+x^2+x+3$ **2.** $y=2x^2+3x+2$ **3.** $y=x^4+\frac{1}{3}x^2+2$

4. $y=\frac{1}{3}x^3-\frac{1}{2}x^2+1$ **5.** $y=x^2+\frac{1}{x}-1$ **6.** $y=\frac{1}{3}x^{1.5}-8$

7. $y=4x-x^2$ **8.** $y=x^3-5x^2+4x$ **9.** $y=81-x^4$

10. $y=x^5-x^3$ **11.** $y=10x^3-10x$ **12.** $y=4\sqrt{x}-1$

B. **1.** $y=x^3+x^2+x+3$ **2.** $y=2x^2+3x+2$ **3.** $y=x^4+\frac{1}{3}x^2+2$

4. $y=\frac{1}{3}x^3-\frac{1}{2}x^2+1$ **5.** $y=x^2+\frac{1}{x}-1\,{}^1/_2$ **6.** $y=\frac{1}{3}x^{1.5}-8$

7. $y=4x-x^2$ **8.** $y=x^3-5x^2+4x$ **9.** $y=81-x^4$

10. $y=x^5-x^3$ **11.** $y=10x^3-10x$ **12.** $y=4\sqrt{x}-1$

Page 43. Integration 2.

A. **1.** $\frac{7}{3}$ **2.** 20 **3.** $\frac{1}{4}$ **4.** 7.5 **5.** $-{}^4/_3$ **6.** 2 **7.** $-\frac{1}{6}$

8. 0 **9.** 72 **10.** $-\frac{1}{6}$

B. **1.** $10\frac{2}{3}$ **2.** $10\frac{2}{3}$ **3.** $11\frac{1}{4}$ **4.** $1\frac{1}{3}$ **5.** $388\frac{4}{5}$ **6.** $\frac{4}{3}$

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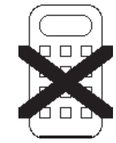
Worksheet D – Matrices (and answers)

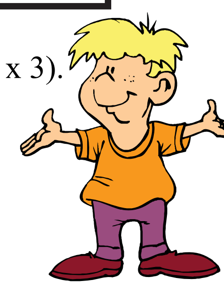
Matrices 1.



Addition, Subtraction and Multiples.

A matrix is a **rectangular** array of numbers.
Each number in the matrix is known as an **element** of the matrix.
The "size" of the matrix, determined by the number of its **rows** and **columns**, is called the **order** of the matrix.

A matrix that has 2 rows and 3 columns has an order "2 by 3" (written as 2 x 3).
Note that the **first** number in the order is always the number of **rows**.



	Matrix	Order	
Example 1	$\begin{pmatrix} 5 & 6 & 7 \\ 3 & 2 & 1 \end{pmatrix}$	2 x 3	
Example 2	$\begin{pmatrix} 6 \\ 4 \end{pmatrix}$	2 x 1	sometimes called a column matrix
Example 3	(-2, 3)	1 x 2	sometimes called a row matrix
Example 4	$\begin{pmatrix} 6 & 5 \\ 7 & 8 \end{pmatrix}$	2 x 2	sometimes called a square matrix of order 2

Question 1 Write down the order of each of the following matrices.

(a) $\begin{pmatrix} 1 & -1 \\ -1 & 2 \\ 0 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 7 & 8 & 9 \end{pmatrix}$ (d) $\begin{pmatrix} 12 \end{pmatrix}$ (e) $\begin{pmatrix} 7 & 5 & 3 & 1 \end{pmatrix}$

Matrix Addition is the process whereby two matrices **of the same order** are combined by **adding together corresponding elements**.

Example 5 Given $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

To find **A** + **B**, add the first elements (1 + 5 = 6)
then add the second elements (2 + (-3) = -1)

so $$\mathbf{A}+\mathbf{B} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 1+5 \\ 2+(-3) \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

Example 6 Given $\mathbf{A}=\begin{pmatrix}3 & 5\\0 & -1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}2 & -4\\7 & 1\end{pmatrix}$

then $\mathbf{A}+\mathbf{B}=\begin{pmatrix}3+2 & 5+(-4)\\0+7 & (-1)+1\end{pmatrix}=\begin{pmatrix}5 & 1\\7 & 0\end{pmatrix}$

Question 2 Express each of the following as a single matrix.

(a) $\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}3\\-5\end{pmatrix}$ (b) $\begin{pmatrix}0 & 1\\-2 & 3\end{pmatrix}+\begin{pmatrix}1 & 0\\-3 & 2\end{pmatrix}$ (c) $\begin{pmatrix}3 & 2 & -1\end{pmatrix}+\begin{pmatrix}-3 & 0 & 1\end{pmatrix}$

(d) $\begin{pmatrix}3 & 4\end{pmatrix}+\begin{pmatrix}2 & -2\end{pmatrix}$ (e) $\begin{pmatrix}2\\1\end{pmatrix}+\begin{pmatrix}-3\\5\end{pmatrix}$ (f) $\begin{pmatrix}6 & -3\end{pmatrix}+\begin{pmatrix}-1 & 2\end{pmatrix}$

(g) $\begin{pmatrix}7\\6\\4\end{pmatrix}+\begin{pmatrix}7\\6\\4\end{pmatrix}$ (h) $\begin{pmatrix}3 & 2\\-1 & 0\\7 & 5\end{pmatrix}+\begin{pmatrix}0 & 2\\5 & -1\\-4 & 2\end{pmatrix}$ (i) $\begin{pmatrix}0 & 1\\-2 & 3\end{pmatrix}+\begin{pmatrix}3 & 0\\-3 & 6\end{pmatrix}$

(j) $\begin{pmatrix}7 & 5 & 3\\2 & -4 & -6\end{pmatrix}+\begin{pmatrix}0 & 2 & -4\\4 & -2 & 6\end{pmatrix}$ (k) $\begin{pmatrix}4 & 8 & 3\\-4 & -4 & -6\end{pmatrix}+\begin{pmatrix}1 & -2 & -4\\4 & -2 & 6\end{pmatrix}$

> The process of matrix addition can be extended to combining more than two matrices, provided **all the matrices involved have the same order**.

Example 7 Given $A=\begin{pmatrix}3 & 5\\0 & -1\end{pmatrix}$, $B=\begin{pmatrix}2 & -4\\7 & 1\end{pmatrix}$ and $C=\begin{pmatrix}-1 & 2\\2 & 0\end{pmatrix}$

then $\mathbf{A}+\mathbf{B}+\mathbf{C}=\begin{pmatrix}3+2+(-1) & 5+(-4)+2\\0+7+2 & (-1)+1+0\end{pmatrix}=\begin{pmatrix}4 & 3\\9 & 0\end{pmatrix}$

Question 3 Express each of the following as a single matrix.

(a) $\begin{pmatrix}0 & 1\\-2 & 3\end{pmatrix}+\begin{pmatrix}1 & 0\\-3 & 2\end{pmatrix}+\begin{pmatrix}2 & 2\\2 & 2\end{pmatrix}$ (b) $\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}3\\-5\end{pmatrix}+\begin{pmatrix}4\\2\end{pmatrix}+\begin{pmatrix}-1\\1\end{pmatrix}$

(c) $\begin{pmatrix}5 & 4 & 3\end{pmatrix}+\begin{pmatrix}2 & 3 & 4\end{pmatrix}+\begin{pmatrix}2 & 2 & 2\end{pmatrix}$ (d) $\begin{pmatrix}5 & -2\end{pmatrix}+\begin{pmatrix}3 & 4\end{pmatrix}+\begin{pmatrix}-1 & 1\end{pmatrix}+\begin{pmatrix}0 & 3\end{pmatrix}$

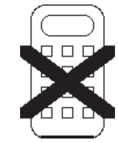
> **Matrix Subtraction** is performed in the same manner as addition, except that the corresponding elements are subtracted. **The matrices must have the same order**.

Example 8 Given $\mathbf{A}=\begin{pmatrix}1\\2\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}5\\-3\end{pmatrix}$

To find **A - B**, then subtract the first elements $(1 - 5 = -4)$

subtract the second elements $(2 - (-3) = 5)$

so $\mathbf{A}-\mathbf{B}=\begin{pmatrix}1\\2\end{pmatrix}-\begin{pmatrix}5\\-3\end{pmatrix}=\begin{pmatrix}1-5\\2-(-3)\end{pmatrix}=\begin{pmatrix}-4\\5\end{pmatrix}$



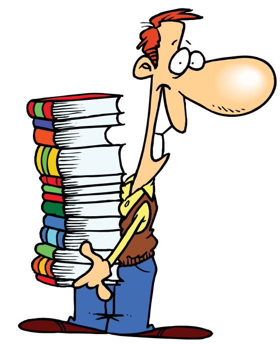
Matrices 2.

Example 9 Given $\mathbf{A}=\begin{pmatrix}3 & 5\\ 0 & -1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}2 & -4\\ 7 & 1\end{pmatrix}$

then $\mathbf{A}-\mathbf{B}=\begin{pmatrix}3-2 & 5-(-4)\\ 0-7 & (-1)-1\end{pmatrix}=\begin{pmatrix}1 & 9\\ -7 & -2\end{pmatrix}$

and $\mathbf{B}-\mathbf{A}=\begin{pmatrix}2-3 & (-4)-5\\ 7-0 & 1-(-1)\end{pmatrix}=\begin{pmatrix}-1 & -9\\ 7 & 2\end{pmatrix}$

Note that **A** - **B** is different from **B** - **A**.



Example 10 Given $\mathbf{A}=\begin{pmatrix}3 & 5\\ 0 & -1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}7\\ 3\end{pmatrix}$

then it is not possible to find **A** - **B** because **A** and **B** have different orders.

Question 4 Express each of the following as a single matrix.

(a) $\begin{pmatrix}1\\ 2\end{pmatrix}-\begin{pmatrix}3\\ -5\end{pmatrix}$ (b) $\begin{pmatrix}0 & 1\\ -2 & 3\end{pmatrix}-\begin{pmatrix}1 & 0\\ -3 & 2\end{pmatrix}$ (c) $\begin{pmatrix}3 & 2 & -1\end{pmatrix}-\begin{pmatrix}-3 & 0 & 1\end{pmatrix}$

(d) $\begin{pmatrix}3 & 4\end{pmatrix}-\begin{pmatrix}2 & -2\end{pmatrix}$ (e) $\begin{pmatrix}5\\ 2\end{pmatrix}-\begin{pmatrix}1\\ -5\end{pmatrix}$ (f) $\begin{pmatrix}4 & -3\end{pmatrix}-\begin{pmatrix}-1 & 2\end{pmatrix}$

(g) $\begin{pmatrix}7\\ 6\\ 4\end{pmatrix}-\begin{pmatrix}7\\ 6\\ 4\end{pmatrix}$ (h) $\begin{pmatrix}3 & 2\\ -1 & 0\\ 7 & 5\end{pmatrix}-\begin{pmatrix}0 & 2\\ 5 & -1\\ -4 & 2\end{pmatrix}$ (i) $\begin{pmatrix}0 & 1\\ -2 & 5\end{pmatrix}-\begin{pmatrix}4 & 0\\ -3 & 4\end{pmatrix}$

(j) $\begin{pmatrix}7 & 5 & 3\\ 2 & -4 & -6\end{pmatrix}-\begin{pmatrix}0 & 2 & -4\\ 4 & -2 & 6\end{pmatrix}$ (k) $\begin{pmatrix}4 & 8 & 3\\ -4 & -4 & -6\end{pmatrix}-\begin{pmatrix}1 & -2 & -4\\ 4 & -2 & 6\end{pmatrix}$

> The processes of matrix addition and subtraction can be mixed, just as in ordinary arithmetic, provided **all the matrices involved have the same order**.

Example 11 Given $\mathbf{A}=\begin{pmatrix}3 & 5\\ 0 & -1\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}2 & -4\\ 7 & 1\end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix}-1 & 2\\ 2 & 0\end{pmatrix}$

then $\mathbf{A}-\mathbf{B}+\mathbf{C}=\begin{pmatrix}3-2+(-1) & 5-(-4)+2\\ 0-7+2 & (-1)-1+0\end{pmatrix}=\begin{pmatrix}0 & 11\\ -5 & -2\end{pmatrix}$

Question 5 Express each of the following as a single matrix.

(a) $\begin{pmatrix}1\\ 2\end{pmatrix}-\begin{pmatrix}3\\ -5\end{pmatrix}+\begin{pmatrix}-1\\ 2\end{pmatrix}$ (b) $\begin{pmatrix}0 & 1\\ -2 & 3\end{pmatrix}+\begin{pmatrix}1 & 0\\ -3 & 2\end{pmatrix}-\begin{pmatrix}2 & -4\\ -3 & 1\end{pmatrix}$

(c) $\begin{pmatrix}3 & 4\end{pmatrix}+\begin{pmatrix}2 & -2\end{pmatrix}-\begin{pmatrix}4 & 1\end{pmatrix}$ (d) $\begin{pmatrix}3 & 2 & -1\end{pmatrix}-\begin{pmatrix}-3 & 0 & 1\end{pmatrix}+\begin{pmatrix}2 & 2 & 2\end{pmatrix}$

> Adding a matrix to itself has the same effect as multiplying each element of the matrix by 2.

Example 12 Given $\mathbf{A}=\begin{pmatrix}3 & 2\\ -4 & 1\end{pmatrix}$

then $\mathbf{A}+\mathbf{A}=\begin{pmatrix}3 & 2\\ -4 & 1\end{pmatrix}+\begin{pmatrix}3 & 2\\ -4 & 1\end{pmatrix}=\begin{pmatrix}6 & 4\\ -8 & 2\end{pmatrix}$

so $2\mathbf{A}=\begin{pmatrix}2\times 3 & 2\times 2\\ 2\times(-4) & 2\times 1\end{pmatrix}=\begin{pmatrix}6 & 4\\ -8 & 2\end{pmatrix}$

> To multiply a matrix by any number we multiply each element of the matrix by that number.

Example 13 Given $\mathbf{A}=\begin{pmatrix}3 & 2\\ -4 & 1\end{pmatrix}$

then $4\mathbf{A}=\begin{pmatrix}4\times 3 & 4\times 2\\ 4\times(-4) & 4\times 1\end{pmatrix}=\begin{pmatrix}12 & 8\\ -16 & 4\end{pmatrix}$

and $-2\mathbf{A}=\begin{pmatrix}(-2)\times 3 & (-2)\times 2\\ (-2)\times(-4) & (-2)\times 1\end{pmatrix}=\begin{pmatrix}-6 & -4\\ 8 & -2\end{pmatrix}$

Question 6 Express each of the following as a single matrix.

(a) $3\begin{pmatrix}-1\\ 5\end{pmatrix}$ (b) $7\begin{pmatrix}4 & -6\\ -5 & 2\end{pmatrix}$ (c) $-5\begin{pmatrix}-9 & 5\end{pmatrix}$ (d) $-4\begin{pmatrix}-2 & 4 & -6\\ 9 & -5 & 3\end{pmatrix}$

(e) $\frac{1}{2}\begin{pmatrix}-6 & 18\\ 24 & 36\end{pmatrix}$ (f) $\frac{1}{3}\begin{pmatrix}-6 & 18\\ 24 & 36\end{pmatrix}$ (g) $\frac{2}{3}\begin{pmatrix}6 & 12\\ 9 & -3\end{pmatrix}$ (h) $1.5\begin{pmatrix}-6 & 18\\ 24 & 36\end{pmatrix}$

> The process of multiplying a matrix by any number can be mixed with the processes of addition and subtraction, provided **all the matrices involved have the same order**.

Example 14 Given $\mathbf{A}=\begin{pmatrix}3 & 5\\ 0 & -1\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}2 & -4\\ 7 & 1\end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix}-1 & 2\\ 2 & 0\end{pmatrix}$

then $\mathbf{A}+2\mathbf{B}-\mathbf{C}=\begin{pmatrix}3+2\times 2-(-1) & 5+2\times(-4)-2\\ 0+2\times 7-2 & (-1)+2\times 1-0\end{pmatrix}=\begin{pmatrix}8 & -5\\ 12 & 1\end{pmatrix}$

Question 7 Express each of the following as a single matrix.

(a) $\begin{pmatrix}1\\ 2\end{pmatrix}+2\begin{pmatrix}3\\ -5\end{pmatrix}$ (b) $\begin{pmatrix}0 & 1\\ -2 & 3\end{pmatrix}-2\begin{pmatrix}1 & 0\\ -3 & 2\end{pmatrix}$ (c) $3\begin{pmatrix}3 & 2 & -1\end{pmatrix}-\begin{pmatrix}-3 & 0 & 1\end{pmatrix}$

(d) $2\begin{pmatrix}3 & 4\end{pmatrix}-3\begin{pmatrix}2 & -2\end{pmatrix}$ (e) $\begin{pmatrix}5\\ 2\end{pmatrix}-2\begin{pmatrix}1\\ -5\end{pmatrix}+\begin{pmatrix}6\\ -1\end{pmatrix}$ (f) $2\begin{pmatrix}4 & -3\end{pmatrix}-3\begin{pmatrix}-1 & 2\end{pmatrix}+\begin{pmatrix}2 & -3\end{pmatrix}$

Matrices 3.

> Numerical information is often written in rows and columns.
> A matrix can be used to hold this type of information.

Example 15
Part of a football league table, showing the results of the *home* and *away* matches of two teams, partway through the season, is given below.

	Home				Away			
Team	Played	Won	Lost	Drawn	Played	Won	Lost	Drawn
United	10	8	0	2	9	4	2	3
Rovers	8	4	2	2	10	5	5	0

The results of the *home* matches can be stored in a 2 x 4 matrix **A**, where $\mathbf{A}=\begin{pmatrix}10 & 8 & 0 & 2\\ 8 & 4 & 2 & 2\end{pmatrix}$.

The results of the *away* matches can be stored in a 2 x 4 matrix **B**, where $\mathbf{B}=\begin{pmatrix}9 & 4 & 2 & 3\\ 10 & 5 & 5 & 0\end{pmatrix}$.

The matrix given by $\mathbf{A}+\mathbf{B}=\begin{pmatrix}10 & 8 & 0 & 2\\ 8 & 4 & 2 & 2\end{pmatrix}+\begin{pmatrix}9 & 4 & 2 & 3\\ 10 & 5 & 5 & 0\end{pmatrix}=\begin{pmatrix}19 & 12 & 2 & 5\\ 18 & 9 & 7 & 2\end{pmatrix}$

shows the results of **all** the matches played.
Thus United has won 12 of the 19 matches it has played altogether.

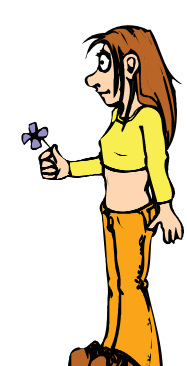
Example 16
A tiler has some tiles, as given in matrix **T**.
The number of these tiles he uses, per square metre, on a particular job, is given by matrix **M.**

$$\mathbf{T}=\begin{array}{c}\begin{array}{cc}\text{Plain} & \text{Patterned}\end{array}\\ \begin{pmatrix}500 & 200\\ 300 & 350\end{pmatrix}\end{array}\begin{array}{l}\\ \text{White}\\ \text{Blue}\end{array} \qquad \mathbf{M}=\begin{array}{c}\begin{array}{cc}\text{Plain} & \text{Patterned}\end{array}\\ \begin{pmatrix}54 & 20\\ 10 & 16\end{pmatrix}\end{array}\begin{array}{l}\\ \text{White}\\ \text{Blue}\end{array}$$

Then the matrix $\mathbf{T}-5\mathbf{M}=\begin{pmatrix}500 & 200\\ 300 & 350\end{pmatrix}-5\begin{pmatrix}54 & 20\\ 10 & 16\end{pmatrix}=\begin{pmatrix}500-270 & 200-100\\ 300-50 & 350-80\end{pmatrix}=\begin{pmatrix}230 & 100\\ 250 & 270\end{pmatrix}$

shows the number of tiles, of each type, he has left, after tiling an area of 5m^2.
Thus he started with 500 plain white tiles, used 5x54 (= 270), and so has 230 left.

Question 8
Some children were asked to name their favourite colour.
Matrix **B** shows the boys' results and **G** shows the girls' results.



$$\mathbf{B}=\begin{array}{c}\begin{array}{ccc}\text{red} & \text{blue} & \text{green}\end{array}\\ \begin{pmatrix}8 & 7 & 6\end{pmatrix}\end{array} \qquad \mathbf{G}=\begin{array}{c}\begin{array}{ccc}\text{red} & \text{blue} & \text{green}\end{array}\\ \begin{pmatrix}6 & 9 & 7\end{pmatrix}\end{array}$$

(a) Find the matrix **B** + **G**.
(b) Describe, in words, the information shown by **B** + **G**.
(c) Which was the most popular colour?

Question 9

Bill recorded the numbers of cars, buses and lorries that passed him, going up and down a road. His results are shown in the two matrices.
Matrix **A** shows the numbers for the first half-hour.
Matrix **B** shows the numbers for the second half-hour.

$$\begin{array}{cc} & \begin{array}{ccc}\text{cars} & \text{buses} & \text{lorries}\end{array} \\ \mathbf{A}= & \begin{pmatrix}25 & 5 & 10\\ 62 & 11 & 7\end{pmatrix}\end{array}\begin{array}{l}\\ \text{up}\\ \text{down}\end{array} \qquad \begin{array}{cc} & \begin{array}{ccc}\text{cars} & \text{buses} & \text{lorries}\end{array} \\ \mathbf{B}= & \begin{pmatrix}33 & 5 & 12\\ 75 & 14 & 6\end{pmatrix}\end{array}\begin{array}{l}\\ \text{up}\\ \text{down}\end{array}$$

(a) (i) Find the matrix **A** + **B.**
(ii) What information does this matrix show?
(iii) In which direction did more lorries travel during the hour Bill was recording his results?

(b) Bill wanted to know how many more cars, buses and lorries passed him, in each direction, during the second half-hour than during the first half-hour.
Form a matrix that shows this information.

Question 10

A supermarket records details of its stock in a row matrix, which is stored in a computer. Part of this matrix, recording some tins of vegetables, is given in matrix **A**.

$$\begin{array}{cc} & \begin{array}{ccc}\text{peas} & \text{beans} & \text{carrots}\end{array} \\ \mathbf{A}= & \begin{pmatrix}43 & 86 & 28\end{pmatrix}\end{array}$$

$$\mathbf{P}_1 = \begin{pmatrix}1 & 0 & 0\end{pmatrix} \qquad \mathbf{P}_2 = \begin{pmatrix}0 & 1 & 0\end{pmatrix} \qquad \mathbf{P}_3 = \begin{pmatrix}0 & 0 & 1\end{pmatrix}$$

Each time a customer buys one of these tins, the corresponding **P** matrix is subtracted from **A** when the bar-code of the tin is read.
Peter bought some tins of vegetables.
In response, the computer calculated $\mathbf{A} - 3\mathbf{P}_1 - \mathbf{P}_2 - 2\mathbf{P}_3$.

(a) Write down the matrix formed by $\mathbf{A} - 3\mathbf{P}_1 - \mathbf{P}_2 - 2\mathbf{P}_3$.
(b) How many tins of each vegetable did Peter buy?

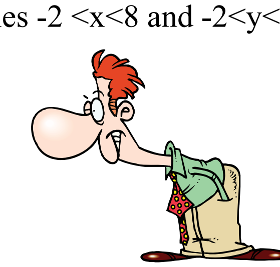
Question 11

The coordinates of triangle *ABC* are shown in the matrix **P**.

$$\begin{array}{cc} & \begin{array}{ccc}A & B & C\end{array} \\ \mathbf{P}= & \begin{pmatrix}2 & 3 & 1\\ 0 & 1 & 2\end{pmatrix}\end{array}$$

Plot the triangle on squared paper using axes -2 <x<8 and -2<y<7.

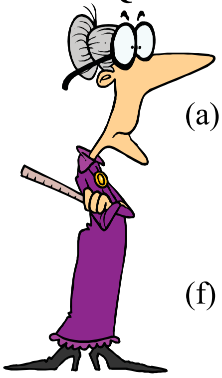
$$\mathbf{Q} = \begin{pmatrix}-1 & -1 & -1\\ 2 & 2 & 2\end{pmatrix}$$



(a) Find the matrix **P** + **Q**.
(b) Find the matrix 2**P**.
(c) On your squared paper add the triangles whose coordinates are given by the matrices
(i) **P** + **Q,** (ii) 2**P**.
(d) What is the relationship between the triangle whose coordinates are given by the matrix **P** + **Q** and triangle *ABC*?
(e) What is the relationship between the triangle whose coordinates are given by the matrix 2**P** and triangle *ABC*?

Matrices 4.

Question 12 Write down the order of each of the following matrices.



(a) $\begin{pmatrix} 5 & 6 & 7 \\ 3 & 2 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 7 & 8 & 9 \end{pmatrix}$ (c) $\begin{pmatrix} 6 & 5 \\ 7 & 8 \end{pmatrix}$ (d) $\begin{pmatrix} 7 \\ 5 \\ 9 \end{pmatrix}$ (e) $\begin{pmatrix} 4 & 7 \end{pmatrix}$

(f) $\begin{pmatrix} 9 & 8 & 7 & 6 \\ 8 & 7 & 6 & 5 \\ 7 & 6 & 5 & 4 \end{pmatrix}$ (g) $\begin{pmatrix} 5 & 8 & 2 \\ 4 & 1 & 6 \\ 3 & 8 & 6 \end{pmatrix}$ (h) $\begin{pmatrix} 25 \end{pmatrix}$ (i) $\begin{pmatrix} 3 & 4 \\ 5 & 4 \\ 5 & 6 \end{pmatrix}$ (j) $\begin{pmatrix} 6 \\ 4 \end{pmatrix}$

Question 13 Express each of the following as a single matrix.

(a) $\begin{pmatrix} 5 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & -3 \end{pmatrix} + 4\begin{pmatrix} -1 & 3 \end{pmatrix}$ (c) $3\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -5 \end{pmatrix}$

(d) $3\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$ (e) $\begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ -3 & 4 \end{pmatrix}$ (f) $\begin{pmatrix} 6 & -3 \end{pmatrix} - 2\begin{pmatrix} -1 & 2 \end{pmatrix}$

(g) $2\begin{pmatrix} 5 \\ 2 \end{pmatrix} + 3\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ (h) $3\begin{pmatrix} 5 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ (i) $2\begin{pmatrix} -3 \\ 2 \end{pmatrix} + 3\begin{pmatrix} 2 \\ -5 \end{pmatrix}$

(j) $2\begin{pmatrix} 6 & -3 \end{pmatrix} + 4\begin{pmatrix} -1 & 2 \end{pmatrix}$ (k) $2\begin{pmatrix} 6 & -3 \end{pmatrix} + 5\begin{pmatrix} -1 & 2 \end{pmatrix}$ (l) $3\begin{pmatrix} 6 & -3 \end{pmatrix} - 6\begin{pmatrix} -1 & 2 \end{pmatrix}$

(m) $4\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$ (n) $3\begin{pmatrix} 0 & 3 \\ -2 & 2 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ -3 & -2 \end{pmatrix}$

(o) $2\begin{pmatrix} 1 & -2 & 4 \\ 3 & 0 & -2 \end{pmatrix} + \begin{pmatrix} 5 & 4 & -3 \\ 1 & -3 & 5 \end{pmatrix}$ (p) $3\begin{pmatrix} 1 & -2 & 4 \\ 3 & 0 & -2 \end{pmatrix} - 2\begin{pmatrix} 5 & 4 & -3 \\ 1 & -3 & 5 \end{pmatrix}$

(q) $\begin{pmatrix} 0 & 1 \\ -3 & 2 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} + 2\begin{pmatrix} 5 & 4 \\ -1 & 3 \end{pmatrix}$ (r) $4\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix} + 2\begin{pmatrix} 5 & 4 \\ -1 & 3 \end{pmatrix}$

(s) $3\begin{pmatrix} 0 & 1 \\ -3 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} - 2\begin{pmatrix} 5 & 4 \\ -1 & 3 \end{pmatrix}$ (t) $4\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix} - 2\begin{pmatrix} 5 & 4 \\ -1 & 3 \end{pmatrix}$

Question 14 Given that $\mathbf{A} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ $\mathbf{C} = \begin{pmatrix} -2 \\ -6 \end{pmatrix}$

express each of the following as a single matrix.

(a) $3\mathbf{B}$ (b) $^1/_2\mathbf{C}$ (c) $-\mathbf{A}$ (d) $\mathbf{A} + 2\mathbf{B}$ (e) $\mathbf{B} + \mathbf{C}$
(f) $\mathbf{A} + \mathbf{B} - \mathbf{C}$ (g) $2\mathbf{A} - 3\mathbf{B}$ (h) $\mathbf{B} + {}^1/_2\mathbf{C}$ (i) $\mathbf{A} - 2\mathbf{B} + 3\mathbf{C}$

Question 15 Given that $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 0 & 4 \\ 1 & -1 \end{pmatrix}$ $\mathbf{C} = \begin{pmatrix} -2 & 0 \\ 6 & -2 \end{pmatrix}$,

express each of the following as a single matrix.

(a) $2\mathbf{A}$ (b) $^1/_2\mathbf{C}$ (c) $-\mathbf{B}$ (d) $\mathbf{A} - 2\mathbf{B}$ (e) $\mathbf{B} - \mathbf{C}$
(f) $\mathbf{A} + \mathbf{B} + \mathbf{C}$ (g) $2\mathbf{A} + 3\mathbf{B}$ (h) $\mathbf{B} - {}^1/_2\mathbf{C}$ (i) $\mathbf{A} + 2\mathbf{B} - 3\mathbf{C}$

Question 16 Given that

$$\mathbf{A} = \begin{pmatrix} 2 & 4 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad \mathbf{F} = \begin{pmatrix} -4 & 6 \end{pmatrix}$$

say whether or not each of the following combinations of matrices can be formed and evaluate those which can.

(a) $\mathbf{A} + \mathbf{B}$ (b) $\mathbf{B} + \mathbf{C}$ (c) $\mathbf{C} - \mathbf{D}$ (d) $2\mathbf{A} + \mathbf{F}$ (e) $2\mathbf{B} + \mathbf{C}$
(f) $2\mathbf{B} - 3\mathbf{E}$ (g) ${}^{1}/_{3}\mathbf{B} + \mathbf{E}$ (h) $\mathbf{C} - \mathbf{D} + \mathbf{F}$ (i) $\mathbf{D} + 2\mathbf{E}$
(j) $2\mathbf{A} + \mathbf{D}$ (k) ${}^{1}/_{2}\mathbf{F} - \mathbf{A}$ (l) $2\mathbf{B} - 2\mathbf{E} + \mathbf{D}$ (m) ${}^{1}/_{2}\mathbf{C} + \mathbf{D}$

Question 17 Find the matrix **X** which satisfies the following equations.

(a) $\mathbf{X} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$ (b) $\mathbf{X} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$ (c) $\mathbf{X} + 3\begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$

(d) $3\mathbf{X} = \begin{pmatrix} -6 \\ 9 \end{pmatrix}$ (e) $3\mathbf{X} = \begin{pmatrix} \frac{1}{2} \\ 2 \end{pmatrix}$ (f) $2\mathbf{X} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$

(g) $\begin{pmatrix} 2 \\ 5 \end{pmatrix} + \mathbf{X} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ (h) $\begin{pmatrix} -2 \\ 3 \end{pmatrix} + 3\mathbf{X} = \mathbf{X}$ (i) $5\mathbf{X} - \begin{pmatrix} 12 \\ -8 \end{pmatrix} = 3\mathbf{X}$

(j) $\begin{pmatrix} 5 \\ 2 \end{pmatrix} - \mathbf{X} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ (k) $\begin{pmatrix} 3 \\ -2 \end{pmatrix} - 2\mathbf{X} = \mathbf{X}$ (l) $\mathbf{X} - \begin{pmatrix} 8 \\ -12 \end{pmatrix} = 5\mathbf{X}$

(m) $3\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \mathbf{X} = 2\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ (n) $2\mathbf{X} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3\begin{pmatrix} 7 \\ -4 \end{pmatrix}$ (o) $4\begin{pmatrix} 5 \\ 2 \end{pmatrix} - \mathbf{X} = 5\begin{pmatrix} 4 \\ -3 \end{pmatrix}$

Question 18 Find the values of x and y from the following equations.

(a) $\begin{pmatrix} 4x & 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 2x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$ (c) $2\begin{pmatrix} x \\ 3y \end{pmatrix} = \begin{pmatrix} -5 \\ 12 \end{pmatrix}$

(d) $\begin{pmatrix} 2x & 6 \end{pmatrix} = \begin{pmatrix} -4 & -3y \end{pmatrix}$ (e) $2\begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} y & 12 \end{pmatrix}$ (f) $\begin{pmatrix} 1 \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ 2 \end{pmatrix}$

(g) $3\begin{pmatrix} x & 4 \\ 6 & y \end{pmatrix} = \begin{pmatrix} 15 & 12 \\ 18 & 6 \end{pmatrix}$ (h) $\begin{pmatrix} -8 & 12 \\ x & 32 \end{pmatrix} = 4\begin{pmatrix} -2 & 3 \\ y & x \end{pmatrix}$ (i) $2\begin{pmatrix} 6 & x \\ 3y & -9 \end{pmatrix} = 3\begin{pmatrix} 4 & x \\ 2y & y \end{pmatrix}$

(j) $x\begin{pmatrix} 2 \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 1 \end{pmatrix}$ (k) $\begin{pmatrix} 3x - y \\ x + 2y \end{pmatrix} = \begin{pmatrix} 17 \\ 1 \end{pmatrix}$ (l) $\begin{pmatrix} 3x \\ x \end{pmatrix} - \begin{pmatrix} y \\ 2y \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix}$

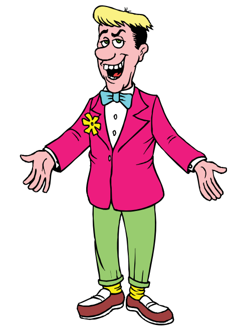
(m) $\begin{pmatrix} 2x - 3 \\ x + 1 \end{pmatrix} = y\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ (n) $x\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2y\begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$ (o) $2\begin{pmatrix} x \\ x - y \end{pmatrix} = 3\begin{pmatrix} y \\ x \end{pmatrix} + \begin{pmatrix} 7 \\ 0 \end{pmatrix}$

Question 19
A carpenter has some screws, of various sizes (given in mm), as shown in matrix **S**.
The numbers of these screws he needs to make one cabinet is shown in matrix **N**.

$$\begin{array}{c} \quad\;\; 20 \quad\; 30 \quad\; 40 \\ \mathbf{S} = \begin{pmatrix} 550 & 400 & 600 \end{pmatrix} \end{array} \qquad \begin{array}{c} \quad\;\; 20 \quad 30 \quad 40 \\ \mathbf{N} = \begin{pmatrix} 32 & 60 & 48 \end{pmatrix} \end{array}$$

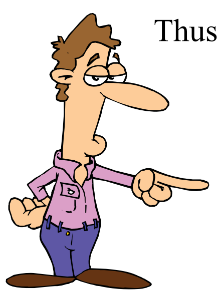
Write the matrix that shows the number of screws, of each size, he has left after making 4 cabinets.

Matrix Multiplication 1.



Example 1 The row matrix $\begin{pmatrix} 2 & 4 & 7 \end{pmatrix}$ and the column matrix $\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$

can be combined by **multiplying**
the **first** element in the row matrix by the **first** element in the column matrix (i.e. 2 x 3),
the **second** element in the row matrix by the **second** element in the column matrix (i.e. 4 x 5),
the **third** element in the row matrix by the **third** element in the column matrix (i.e. 7 x 1),
and **adding** the three **products** together, to give a 1 x 1 matrix.



Thus $$\begin{pmatrix} 2 & 4 & 7 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\times3+4\times5+7\times1 \end{pmatrix} = \begin{pmatrix} 33 \end{pmatrix}.$$

> This method can be used to combine a row matrix and a column matrix, **with the same number of elements** in each, of any size.

Example 2 $$\begin{pmatrix} 5 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 5\times4+3\times(-2) \end{pmatrix} = \begin{pmatrix} 20-6 \end{pmatrix} = \begin{pmatrix} 14 \end{pmatrix}$$

Question 1 Use the method shown in Example 1 and Example 2 to combine the following row and column matrices to give a 1 x 1 matrix.

(a) $\begin{pmatrix} 7 & 2 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 0 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 3 & -4 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$

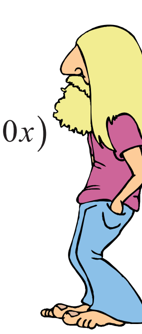
(d) $\begin{pmatrix} 4 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ (e) $\begin{pmatrix} 4 & -5 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ (f) $\begin{pmatrix} -4 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

(g) $\begin{pmatrix} 6 & 5 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 7 \\ 0 \\ 4 \end{pmatrix}$ (h) $\begin{pmatrix} 6 & -5 & -2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -7 \\ 0 \\ 4 \end{pmatrix}$ (i) $\begin{pmatrix} 6 & 5 & 2 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ -7 \\ 0 \\ -4 \end{pmatrix}$

Question 2 Find the value of x in each of the following equations.

(a) $\begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ 4 \end{pmatrix} = \begin{pmatrix} 18 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ -2 \end{pmatrix} = \begin{pmatrix} 34 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ x \end{pmatrix} = \begin{pmatrix} 29 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 2x & 4 \end{pmatrix}\begin{pmatrix} x \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 52 \end{pmatrix}$ (e) $\begin{pmatrix} 3x & 2 & -4 \end{pmatrix}\begin{pmatrix} 6 \\ x \\ 3 \end{pmatrix} = \begin{pmatrix} 48 \end{pmatrix}$ (f) $\begin{pmatrix} 3 & 2x & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 10x \end{pmatrix}$



(g) $\begin{pmatrix} 5 & 3 \end{pmatrix}\begin{pmatrix} -2x \\ x \end{pmatrix} = \begin{pmatrix} 28 \end{pmatrix}$ (h) $\begin{pmatrix} 2 & x \end{pmatrix}\begin{pmatrix} 4 \\ x \end{pmatrix} = \begin{pmatrix} 33 \end{pmatrix}$ (i) $\begin{pmatrix} 5 & x \end{pmatrix}\begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} 6 \end{pmatrix}$

Example 3 The first matrix shows the results of 19 matches played by a football team. The second matrix shows the points awarded for each result.

$$\begin{array}{lccc} & \text{Won} & \text{Lost} & \text{Drawn} \\ \text{Number of matches} & (12 & 2 & 5) \end{array} \qquad \begin{array}{c} \text{Points} \\ \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{each Win} \\ \text{each Loss} \\ \text{each Draw} \end{array}$$

The matrix

$$\begin{pmatrix} 12 & 2 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 12\times3+2\times0+5\times1 \end{pmatrix} = \begin{pmatrix} 41 \end{pmatrix}$$

shows the total points the team was awarded for these matches.

Question 3 Mary bought the materials shown in the first matrix at the prices shown in the second. Form a matrix to find how much she spent.

$$\begin{array}{lccc} & \text{Pens} & \text{Pencils} & \text{Rubbers} \\ \text{Numbers bought} & (1 & 6 & 2) \end{array} \qquad \begin{array}{c} \text{Price in cents} \\ \begin{pmatrix} 200 \\ 50 \\ 40 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{each Pen} \\ \text{each Pencil} \\ \text{each Rubber} \end{array}$$

Question 4 A shop sells 35mm film in sizes of 24 and 36 exposures.
The prices of the films are in the second matrix.
The numbers of each type of film William bought are shown in the first matrix.
Form a matrix to find the total cost of the films he bought.

$$\begin{array}{lcc} & \text{24 exposures} & \text{36 exposures} \\ \text{Numbers bought} & (5 & 3) \end{array} \qquad \begin{array}{c} \text{Price (\$)} \\ \begin{pmatrix} 2.70 \\ 3.85 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{each 24 exposure film} \\ \text{each 36 exposure film} \end{array}$$

Question 5 A farmer has some bags of fertiliser in three different sizes. The numbers of bags are shown in the first matrix. The second matrix shows the mass of each bag.

$$\begin{array}{lccc} & \text{Large} & \text{Medium} & \text{Small} \\ \text{Numbers of bags} & (5 & 6 & 9) \end{array} \qquad \begin{array}{c} \text{Mass (kg)} \\ \begin{pmatrix} 40 \\ 25 \\ 5 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{each Large bag} \\ \text{each Medium bag} \\ \text{each Small bag} \end{array}$$

(a) Form a matrix to find the total mass of all these bags.
(b) Calculate the mean mass of one of these bags.

Question 6 A car went on a journey which was split into three stages.
The average speed, in km/h, of the car on each stage is shown in the first matrix.
The second matrix shows the time, in hours, taken on each stage.

$$\begin{array}{lccc} & \text{First} & \text{Second} & \text{Third} \\ \text{Average speed} & (50 & 30 & 40) \end{array} \qquad \begin{array}{c} \text{Time} \\ \begin{pmatrix} 3 \\ \frac{1}{2} \\ 1\frac{1}{2} \end{pmatrix} \end{array} \begin{array}{l} \\ \text{First} \\ \text{Second} \\ \text{Third} \end{array}$$

(a) Form a matrix to show the total distance travelled.
(b) Calculate the average speed for the whole journey.

Matrix Multiplication 2.

We have seen that

$$\begin{pmatrix} 2 & 4 & 7 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\times3+4\times5+7\times1 \end{pmatrix} = \begin{pmatrix} 33 \end{pmatrix}.$$

We also have

$$\begin{pmatrix} 6 & 8 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 6\times3+8\times5+3\times1 \end{pmatrix} = \begin{pmatrix} 61 \end{pmatrix}.$$

These two results can be combined to give

$$\begin{pmatrix} 2 & 4 & 7 \\ 6 & 8 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\times3+4\times5+7\times1 \\ 6\times3+8\times5+3\times1 \end{pmatrix} = \begin{pmatrix} 33 \\ 61 \end{pmatrix}.$$

Thus we can combine a 2 x 3 matrix and a 3 x 1 matrix to give a 2 x 1 matrix by multiplying elements in the **rows of the first matrix** by the corresponding elements in the **columns of the second matrix** and adding these products.

In a similiar way we have
Example 4

$$\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} \mathbf{6} \\ \mathbf{5} \end{pmatrix} = \begin{pmatrix} 1\times\mathbf{6}+3\times\mathbf{5} \\ 4\times\mathbf{6}+2\times\mathbf{5} \end{pmatrix} = \begin{pmatrix} 21 \\ 34 \end{pmatrix}$$

Thus we can combine a 2 x 2 matrix and a 2 x 1 matrix to give a 2 x 1 matrix by multiplying elements in the **rows of the first matrix** by the corresponding elements in the **columns of the second matrix** and adding these products.

Question 7 Use the method shown above to combine the following matrices to give a 2 x 1 matrix.

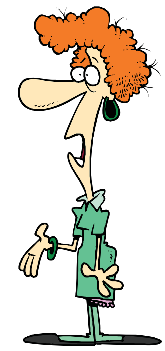
(a) $\begin{pmatrix} 5 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 6 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 6 \\ 2 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 4 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 6 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 4 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 2 \end{pmatrix}$ (e) $\begin{pmatrix} 3 & -2 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} -1 \\ 5 \end{pmatrix}$ (f) $\begin{pmatrix} -3 & 2 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} 5 \\ -1 \end{pmatrix}$

(g) $\begin{pmatrix} 2 & 7 & 4 \\ 5 & 3 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ (h) $\begin{pmatrix} 2 & 7 & 4 \\ 5 & 3 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ (i) $\begin{pmatrix} 5 & 3 & 6 \\ 2 & 7 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

(j) $\begin{pmatrix} 7 & 3 & 6 \\ 2 & 0 & 5 \end{pmatrix}\begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix}$ (k) $\begin{pmatrix} 2 & -4 & 6 \\ 3 & 1 & -2 \end{pmatrix}\begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$ (l) $\begin{pmatrix} 5 & 6 & -4 \\ 8 & -7 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix}$

We have seen that $\begin{pmatrix}2 & 4 & 7\end{pmatrix}\begin{pmatrix}3\\5\\1\end{pmatrix}=\begin{pmatrix}2\times3+4\times5+7\times1\end{pmatrix}=\begin{pmatrix}33\end{pmatrix}$.



We also have $\begin{pmatrix}2 & 4 & 7\end{pmatrix}\begin{pmatrix}6\\2\\4\end{pmatrix}=\begin{pmatrix}2\times6+4\times2+7\times4\end{pmatrix}=\begin{pmatrix}48\end{pmatrix}$.

These two results can be combined to give

$$\begin{pmatrix}2 & 4 & 7\end{pmatrix}\begin{pmatrix}3 & 6\\5 & 2\\1 & 4\end{pmatrix}=\begin{pmatrix}2\times3+4\times5+7\times1 & 2\times6+4\times2+7\times4\end{pmatrix}=\begin{pmatrix}33 & 48\end{pmatrix}.$$

> Thus we can combine a 1 x 3 matrix and a 3 x 2 matrix to give a 1 x 2 matrix by multiplying elements in the **rows of the first matrix** by the corresponding elements in the **columns of the second matrix** and adding these products.

Example 5

$$\begin{pmatrix}\mathbf{2} & \mathbf{6}\end{pmatrix}\begin{pmatrix}7 & 3\\4 & 1\end{pmatrix}=\begin{pmatrix}\mathbf{2}\times7+\mathbf{6}\times4 & \mathbf{2}\times3+\mathbf{6}\times1\end{pmatrix}=\begin{pmatrix}38 & 12\end{pmatrix}$$

> Thus we can combine a 1 x 2 matrix and a 2 x 2 matrix to give a 1 x 2 matrix by multiplying elements in the **rows of the first matrix** by the corresponding elements in the **columns of the second matrix** and adding these products.

Question 8 Use the method shown above to combine the following matrices to give a 1 x 2 matrix.

(a) $\begin{pmatrix}5 & 1\end{pmatrix}\begin{pmatrix}2 & 6\\3 & 4\end{pmatrix}$ (b) $\begin{pmatrix}5 & 1\end{pmatrix}\begin{pmatrix}3 & 4\\2 & 6\end{pmatrix}$ (c) $\begin{pmatrix}1 & 5\end{pmatrix}\begin{pmatrix}3 & 4\\2 & 6\end{pmatrix}$

(d) $\begin{pmatrix}1 & 5\end{pmatrix}\begin{pmatrix}2 & 6\\3 & 4\end{pmatrix}$ (e) $\begin{pmatrix}3 & -2\end{pmatrix}\begin{pmatrix}-4 & 6\\5 & -1\end{pmatrix}$ (f) $\begin{pmatrix}-3 & 2\end{pmatrix}\begin{pmatrix}4 & 6\\-5 & -1\end{pmatrix}$

(g) $\begin{pmatrix}-1 & 5\end{pmatrix}\begin{pmatrix}2 & 6\\3 & 4\end{pmatrix}$ (h) $\begin{pmatrix}5 & -2\end{pmatrix}\begin{pmatrix}-4 & 6\\3 & -1\end{pmatrix}$ (i) $\begin{pmatrix}-3 & 2\end{pmatrix}\begin{pmatrix}-5 & 6\\-4 & -1\end{pmatrix}$

(j) $\begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}2 & 5\\7 & 3\\4 & 6\end{pmatrix}$ (k) $\begin{pmatrix}1 & 8 & 4\end{pmatrix}\begin{pmatrix}2 & 7\\0 & 3\\5 & 6\end{pmatrix}$ (l) $\begin{pmatrix}5 & 0 & 2\end{pmatrix}\begin{pmatrix}6 & -2\\-4 & 1\\2 & 3\end{pmatrix}$

(m) $\begin{pmatrix}3 & -2 & 3\end{pmatrix}\begin{pmatrix}8 & -4\\7 & 6\\-1 & 5\end{pmatrix}$ (n) $\begin{pmatrix}3 & -2 & 3\end{pmatrix}\begin{pmatrix}-4 & 8\\6 & 7\\5 & -1\end{pmatrix}$ (o) $\begin{pmatrix}2 & -2 & -3\end{pmatrix}\begin{pmatrix}8 & -4\\-7 & 6\\-1 & 5\end{pmatrix}$

Matrix Multiplication 3.

So far matrices with a variety of orders have been combined. These examples are summarised in the table below, which shows the orders of the matrices used and the order of the resulting matrix.

First matrix	Second matrix	Resulting matrix
1 x **3**	**3** x 1	1 x 1
1 x **2**	**2** x 1	1 x 1
2 x **3**	**3** x 1	2 x 1
2 x **2**	**2** x 1	2 x 1
1 x **3**	**3** x 2	1 x 2

The process by which these matrices have been combined is known as **matrix multiplication**.

In general, two matrices can be multiplied in this way **provided the number of columns in the first matrix** is equal to **the number of rows in the second matrix**.

The **resulting matrix** has its
number of rows equal to the number of **rows** in the **first** matrix,
number of columns equal to the number of **columns** in the **second** matrix.

Given a matrix **A**, of order p x q, and a matrix **B**, of order r x s
the product **AB** is only possible if q = r and, when q = r, the matrix **AB** has order p x s,
the product **BA** is only possible if s = p and, when s = p, the matrix **BA** has order r x q.
Both products **AB** and **BA** are only possible when **A** and **B** are both square matrices of the same order.

Question 9 Combine these matrices to form a single matrix.

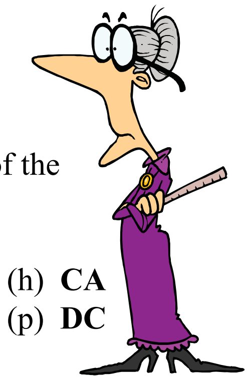
(a) $\begin{pmatrix} 3 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 4 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 5 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & -2 \end{pmatrix}\begin{pmatrix} 2 & -3 & 5 \\ -3 & 4 & -3 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (e) $\begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 2 & 4 \end{pmatrix}$ (f) $\begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -4 & 2 \\ 1 & 3 & -1 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 3 & -2 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ (h) $\begin{pmatrix} 1 & 3 & -2 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ 5 & 3 \\ 6 & 4 \end{pmatrix}$ (i) $\begin{pmatrix} 1 & 3 & -2 \end{pmatrix}\begin{pmatrix} 6 & -2 & 0 \\ 5 & 3 & -1 \\ 4 & 4 & 2 \end{pmatrix}$

(j) $\begin{pmatrix} 1 & -3 & 2 \\ 4 & 2 & -1 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \\ -1 \end{pmatrix}$ (k) $\begin{pmatrix} 1 & -3 & 2 \\ 4 & 2 & -1 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ 3 & 5 \\ 4 & 6 \end{pmatrix}$

Question 10 For each of the parts in Question 9, write down the orders of
(i) the first matrix,
(ii) the second matrix,
(iii) the resulting matrix.



Question 11 The orders of the matrices **A**, **B**, **C** and **D** are given below.
A is 1 x 2, **B** is 2 x 1, **C** is 2 x 2 and **D** is 2 x 3

Say whether each of the following multiplications is possible, and give the order of the resulting matrix in the cases where the multiplication is possible.

(a) $\mathbf{A}^2$ (b) $\mathbf{B}^2$ (c) $\mathbf{C}^2$ (d) $\mathbf{D}^2$ (e) **AB** (f) **BA** (g) **AC** (h) **CA**
(i) **AD** (j) **DA** (k) **BC** (l) **CB** (m) **BD** (n) **DB** (o) **CD** (p) **DC**

Question 12 Given that

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \quad \mathbf{F} = \begin{pmatrix} 6 & -3 \end{pmatrix}$$

perform, where possible, the following products. If any are impossible, then say so.

(a) **AB** (b) **BA** (c) **AC** (d) **CA** (e) **AF** (f) **FA**
(g) **BC** (h) **CB** (i) **BE** (j) **EB** (k) **CD** (l) **DC**

Question 13 Given that

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 4 & -1 & 2 \\ -2 & 0 & 3 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 3 & 4 \\ 0 & 2 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 0 & -1 \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

perform, where possible, the following products. If any are impossible, then say so.

(a) **AB** (b) **BA** (c) **AC** (d) **CA** (e) **AD** (f) **DA**
(g) **AE** (h) **EA** (i) **BD** (j) **DB** (k) **DE** (l) **ED**

Question 14 The results of the matches played by two football teams are given in the first matrix.
The second matrix shows the points awarded for each result.

$$\begin{array}{ll} & \begin{array}{ccc} \text{Won} & \text{Lost} & \text{Drawn} \end{array} \\ \begin{array}{l} \text{United} \\ \text{Rovers} \end{array} & \begin{pmatrix} 12 & 3 & 4 \\ 9 & 2 & 7 \end{pmatrix} \end{array} \qquad \begin{array}{ll} \text{Points} & \\ \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} & \begin{array}{l} \text{each Win} \\ \text{each Loss} \\ \text{each Draw} \end{array} \end{array}$$

(a) Form a matrix which shows the points awarded, to each team, for these matches.
(b) Which team has the most points, and by how many?

Question 15 Mary wishes to buy the items shown in the first matrix.
The second matrix shows the prices of these items in two different shops.

$$\begin{array}{ll} & \begin{array}{ccc} \text{Pens} & \text{Pencils} & \text{Rubbers} \end{array} \\ \text{Numbers bought} & \begin{pmatrix} 1 & 6 & 2 \end{pmatrix} \end{array} \qquad \begin{array}{ll} \text{Price in cents} & \\ \begin{array}{cc} \text{Shop A} & \text{Shop B} \end{array} & \\ \begin{pmatrix} 200 & 180 \\ 50 & 60 \\ 40 & 35 \end{pmatrix} & \begin{array}{l} \text{each Pen} \\ \text{each Pencil} \\ \text{each Rubber} \end{array} \end{array}$$

(a) Form a matrix which shows how much Mary would pay for these items at each shop.
(b) At which shop would Mary pay less, and by how much?

Commutative Matrices.

Question 1 Each part in this question multiplies two matrices.
Parts (i) and (ii) have the same matrices multiplied in a different order.
Multiply the matrices and compare your answers.

(a) (i) $\begin{pmatrix}1 & 2\\ 4 & 3\end{pmatrix}\begin{pmatrix}0 & 3\\ 3 & 5\end{pmatrix}$ (ii) $\begin{pmatrix}0 & 3\\ 3 & 5\end{pmatrix}\begin{pmatrix}1 & 2\\ 4 & 3\end{pmatrix}$ (b) (i) $\begin{pmatrix}3 & 2\\ 2 & 3\end{pmatrix}\begin{pmatrix}1 & 4\\ 4 & 1\end{pmatrix}$ (ii) $\begin{pmatrix}1 & 4\\ 4 & 1\end{pmatrix}\begin{pmatrix}3 & 2\\ 2 & 3\end{pmatrix}$

(c) (i) $\begin{pmatrix}3 & 2\\ 4 & 3\end{pmatrix}\begin{pmatrix}3 & -2\\ -4 & 3\end{pmatrix}$ (ii) $\begin{pmatrix}3 & -2\\ -4 & 3\end{pmatrix}\begin{pmatrix}3 & 2\\ 4 & 3\end{pmatrix}$ (d) (i) $\begin{pmatrix}5 & 5\\ 3 & 3\end{pmatrix}\begin{pmatrix}1 & 2\\ 4 & 3\end{pmatrix}$ (ii) $\begin{pmatrix}1 & 2\\ 4 & 3\end{pmatrix}\begin{pmatrix}5 & 5\\ 3 & 3\end{pmatrix}$

(e) (i) $\begin{pmatrix}3 & -1\\ 2 & 2\end{pmatrix}\begin{pmatrix}2 & 1\\ -2 & 3\end{pmatrix}$ (ii) $\begin{pmatrix}2 & 1\\ -2 & 3\end{pmatrix}\begin{pmatrix}3 & -1\\ 2 & 2\end{pmatrix}$ (f) (i) $\begin{pmatrix}-1 & 0\\ 1 & 2\end{pmatrix}\begin{pmatrix}3 & 0\\ 1 & 2\end{pmatrix}$ (ii) $\begin{pmatrix}3 & 0\\ 1 & 2\end{pmatrix}\begin{pmatrix}-1 & 0\\ 1 & 2\end{pmatrix}$

(g) (i) $\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ -3 & 4\end{pmatrix}$ (ii) $\begin{pmatrix}1 & 0\\ -3 & 4\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}$ (h) (i) $\begin{pmatrix}3 & 0\\ -1 & 4\end{pmatrix}\begin{pmatrix}0 & 2\\ 3 & -1\end{pmatrix}$ (ii) $\begin{pmatrix}0 & 2\\ 3 & -1\end{pmatrix}\begin{pmatrix}3 & 0\\ -1 & 4\end{pmatrix}$

> When **A** and **B** are square matrices of the same order, the products **AB** and **BA** are **not generally equal**.
> Matrices **A** and **B,** such that **AB** = **BA,** are said to be **commutative.**

Question 2 In each of the following, find the value of x for which **AB** = **BA**.

(a) $\mathbf{A} = \begin{pmatrix}5 & 3\\ 0 & -1\end{pmatrix}$ $\mathbf{B} = \begin{pmatrix}x & -3\\ 0 & 5\end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix}5 & 3\\ 0 & -1\end{pmatrix}$ $\mathbf{B} = \begin{pmatrix}25 & x\\ 0 & 1\end{pmatrix}$

(c) $\mathbf{A} = \begin{pmatrix}5 & 3\\ x & 5\end{pmatrix}$ $\mathbf{B} = \begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$ (d) $\mathbf{A} = \begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$ $\mathbf{B} = \begin{pmatrix}5 & 4\\ 4 & x\end{pmatrix}$

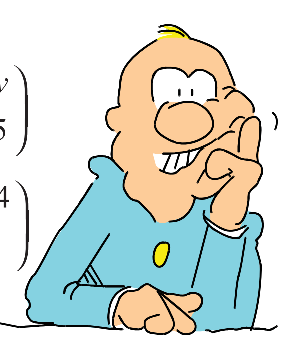
(e) $\mathbf{A} = \begin{pmatrix}5 & 3\\ 3 & x\end{pmatrix}$ $\mathbf{B} = \begin{pmatrix}x & -3\\ -3 & 5\end{pmatrix}$ (f) $\mathbf{A} = \begin{pmatrix}4 & 3\\ 0 & -2\end{pmatrix}$ $\mathbf{B} = \begin{pmatrix}-2 & x\\ 0 & 4\end{pmatrix}$

(g) $\mathbf{A} = \begin{pmatrix}3 & 3\\ 0 & -3\end{pmatrix}$ $\mathbf{B} = \begin{pmatrix}23 & 12\\ 0 & x\end{pmatrix}$ (h) $\mathbf{A} = \begin{pmatrix}2 & 3\\ 0 & x\end{pmatrix}$ $\mathbf{B} = \begin{pmatrix}x & -3\\ 0 & 2\end{pmatrix}$

Question 3 In each of the following, find the values of x and y for which **AB** = **BA**.

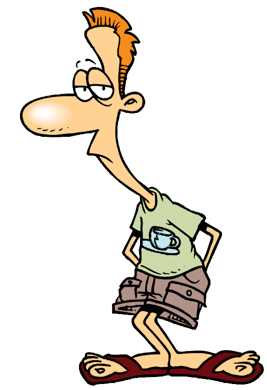
(a) $\mathbf{A} = \begin{pmatrix}3 & x\\ -1 & 4\end{pmatrix}$ $\mathbf{B} = \begin{pmatrix}y & 6\\ 2 & 0\end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix}2 & 0\\ -1 & x\end{pmatrix}$ $\mathbf{B} = \begin{pmatrix}2 & y\\ -3 & 5\end{pmatrix}$

(c) $\mathbf{A} = \begin{pmatrix}1 & 2\\ 1 & x\end{pmatrix}$ $\mathbf{B} = \begin{pmatrix}3 & 4\\ 2 & y\end{pmatrix}$ (d) $\mathbf{A} = \begin{pmatrix}-1 & -6\\ x & 2\end{pmatrix}$ $\mathbf{B} = \begin{pmatrix}y & -4\\ 2 & 3\end{pmatrix}$



Question 4 Given that $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 5 & 4 \\ 2 & 0 \end{pmatrix}$ copy and complete the following.

Hint : in the later parts, use the appropriate answers you obtained in the earlier parts.



(a) $\mathbf{A}^2 = \mathbf{AA} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} .... & .... \\ .... & .... \end{pmatrix}$

(b) $\mathbf{A}^3 = \mathbf{A}^2\,\mathbf{A} = \begin{pmatrix} .... & .... \\ .... & .... \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} .... & .... \\ .... & .... \end{pmatrix}$

(c) $\mathbf{AA}^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} .... & .... \\ .... & .... \end{pmatrix} = \begin{pmatrix} .... & .... \\ .... & .... \end{pmatrix}$

(d) $\mathbf{AB} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 4 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} .... & .... \\ .... & .... \end{pmatrix}$

(e) $\mathbf{BA} = \begin{pmatrix} 5 & 4 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} .... & .... \\ .... & .... \end{pmatrix}$

(f) $\mathbf{A} - \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 4 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} .... & .... \\ .... & .... \end{pmatrix}$

(g) $\mathbf{A}(\mathbf{A} - \mathbf{B}) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} .... & .... \\ .... & .... \end{pmatrix} = \begin{pmatrix} .... & .... \\ .... & .... \end{pmatrix}$

(h) $\mathbf{A}^2 - \mathbf{AB} = \begin{pmatrix} .... & .... \\ .... & .... \end{pmatrix} - \begin{pmatrix} .... & .... \\ .... & .... \end{pmatrix} = \begin{pmatrix} .... & .... \\ .... & .... \end{pmatrix}$

(i) $\mathbf{A}^2 - \mathbf{BA} = \begin{pmatrix} .... & .... \\ .... & .... \end{pmatrix} - \begin{pmatrix} .... & .... \\ .... & .... \end{pmatrix} = \begin{pmatrix} .... & .... \\ .... & .... \end{pmatrix}$

(j) $(\mathbf{A} - \mathbf{B})\mathbf{A} = \begin{pmatrix} .... & .... \\ .... & .... \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} .... & .... \\ .... & .... \end{pmatrix}$

Note that

(i) $\mathbf{A}$ multiplied by $\mathbf{A}$ is written $\mathbf{A}^2$, similiarly for other powers of $\mathbf{A}$,

(ii) $\mathbf{A}(\mathbf{A} - \mathbf{B}) = \mathbf{A}^2 - \mathbf{AB}$ and $(\mathbf{A} - \mathbf{B})\mathbf{A} = \mathbf{A}^2 - \mathbf{BA}$,

(iii) in general $\mathbf{A}(\mathbf{A} - \mathbf{B}) \neq (\mathbf{A} - \mathbf{B})\mathbf{A}$.

> Removal of brackets involving matrices follows the same rules as ordinary algebra,
> **except that the positions of the matrices must not be changed.**

Question 5 Given that $\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ find the following.

Hint : in the later parts, use the appropriate answers you obtained in the earlier parts.

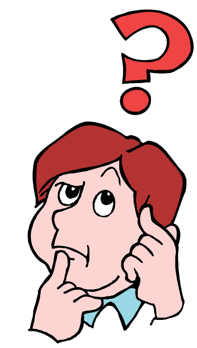
(a) $\mathbf{A}^2$ (b) $\mathbf{B}^3$ (c) $\mathbf{AB}$ (d) $\mathbf{BA}$ (e) $\mathbf{AB} - \mathbf{BA}$

(f) $\mathbf{A}^2 + \mathbf{AB}$ (g) $\mathbf{A}(\mathbf{B} - \mathbf{A})$ (h) $\mathbf{A}^2 + 3\mathbf{A}$ (i) $\mathbf{A}^2 - \mathbf{B}^2$ (j) $(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B})$

(k) $4\mathbf{A}^2 - \mathbf{B}^2$ (l) $3\mathbf{A}^2 + 2\mathbf{B}^2$ (m) $\mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$ (n) $(\mathbf{A} + \mathbf{B})^2$

Identity and Zero Matrices.

Question 1 Each part in this question multiplies two matrices.
Parts (i) and (ii) have the same matrices multiplied in a different order.
Multiply the matrices and compare your answers.



(a) (i) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 2 & 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 5 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) (i) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 7 & 2 \\ 5 & 6 \end{pmatrix}$ (ii) $\begin{pmatrix} 7 & 2 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(c) (i) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(d) (i) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 9 & 8 & 7 \\ 4 & 5 & 6 \\ 3 & 2 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 9 & 8 & 7 \\ 4 & 5 & 6 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Note that, in each part, the answer is identical to one of the matrices.

> The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the **identity matrix** of order two.
>
> The matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is called the **identity matrix** of order three.
>
> Identity matrices are denoted by **I**.
>
> When **A** is a square matrix and **I** is the identity matrix with the same order as **A** then, $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$

Question 2 Given $\mathbf{A} = \begin{pmatrix} 6 & 5 & 4 \\ 1 & 2 & 3 \end{pmatrix}$, write down the orders of the matrices **I** such that

(a) $\mathbf{AI} = \mathbf{A}$ (ii) $\mathbf{IA} = \mathbf{A}$.

Question 3 Given $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ find

(a) $\mathbf{I}^2$ (b) $\mathbf{I}^3$ (c) $\mathbf{I}^4$ (d) $\mathbf{I}^2 + \mathbf{I}^3$ (e) $5\mathbf{I}^4 - 2\mathbf{I}^{10}$

Question 4 $\mathbf{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(a) Find $\mathbf{C}^2$.
(b) Hence find (i) $\mathbf{C}^4$ (ii) $\mathbf{C}^6$ (iii) $\mathbf{C}^{11}$ (iv) $\mathbf{C}^{100}$.

Question 5 Each part in this question multiplies two matrices.
Parts (i) and (ii) have the same matrices multiplied in a different order.
Multiply the matrices and compare your answers.

(a) (i) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 2 & 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 5 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(b) (i) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 7 & 2 \\ 5 & 6 \end{pmatrix}$ (ii) $\begin{pmatrix} 7 & 2 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(c) (i) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(d) (i) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 9 & 8 & 7 \\ 4 & 5 & 6 \\ 3 & 2 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 9 & 8 & 7 \\ 4 & 5 & 6 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Note that, in each part, the answer is identical to one of the matrices.

The matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is called the **zero matrix** of order two.

The matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is called the **zero matrix** of order three.

Identity matrices are denoted by **O**.

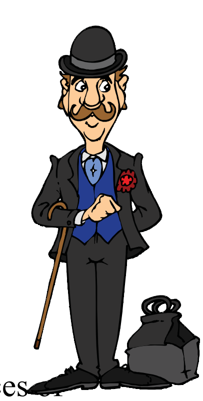
When **A** is a square matrix and **O** is the identity matrix with the same order as **A** then

$$\mathbf{OA} = \mathbf{AO} = \mathbf{O}$$

Question 6 Given $\mathbf{X} = \begin{pmatrix} 2 & 1 \end{pmatrix}$ and $\mathbf{Y} = \begin{pmatrix} 3 & -4 \\ -6 & 8 \end{pmatrix}$.

(a) Show that **XY** = **O**.
(b) When x and y are numbers and $xy = 0$, what can be deduced?
Is a similar result true for matrices?

Question 7 Find the values of x in the following.



(a) $\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} x & 6 \\ 6 & -9 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 \\ 3 & x \end{pmatrix}\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -3 & x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 2 \\ 4 & x \end{pmatrix}\begin{pmatrix} 6 & -2 \\ -4 & 3 \end{pmatrix} = 10\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Question 8 **A** and **B** are 2 x 2 matrices. **I** and **O** are the identity and zero matrices of order 2. Say whether or not each of the following statements is **always** true.
Provide a counter-example in the cases when the statement is **not always** true.

(a) **AO** = **O** (b) **AB** = **O** implies **A** = **O** or **B** = **O** (c) $(\mathbf{A}^3)^4 = \mathbf{A}^{12}$
(d) $(\mathbf{AB})^2 = \mathbf{A}^2\mathbf{B}^2$ (e) $\mathbf{A}^3 \text{ x } \mathbf{A}^5 = \mathbf{A}^8$ (f) **AB** = **I** implies **A** = **I** or **B** = **I**
(g) **AB** = **B** implies **A** = **I**

Multiple Matrix Multiplications.

The process of matrix multiplication can be extended to combine more than two matrices, provided **adjacent matrices have orders which are compatible for multiplication**.

Thus if matrix **A** has order 1x3, matrix **B** has order 3x2 and matrix **C** has order 2x2 then the product **ABC** can be found as (**AB**)**C** or **A**(**BC**), and the order of the product **ABC** is 1x2. (Seen from **1**x[3 3]x[2 2]x**2** .)

Example Given $\mathbf{A} = \begin{pmatrix} 2 & 3 & 4 \end{pmatrix}$ $\quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 2 & 4 \end{pmatrix}$ $\quad \mathbf{C} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$.

Then

$$\mathbf{AB} = \begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 10 & 25 \end{pmatrix}$$

and $$(\mathbf{AB})\mathbf{C} = \begin{pmatrix} 10 & 25 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 115 & 60 \end{pmatrix}$$

Also

$$\mathbf{BC} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 9 & 6 \\ 20 & 10 \end{pmatrix}$$

and $$\mathbf{A}(\mathbf{BC}) = \begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 9 & 6 \\ 20 & 10 \end{pmatrix} = \begin{pmatrix} 115 & 60 \end{pmatrix}$$

Thus **ABC** = (115 60) no matter which way it was obtained.

Question 1 Given

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 3 & 2 & 1 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 3 & 2 \\ 1 & -3 \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix} \quad \mathbf{F} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}.$$

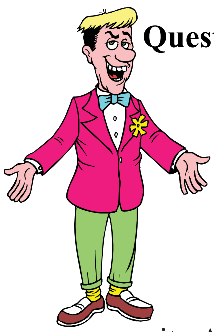
In each of the following cases state whether or not it possible to form the product and, if so, write down the order of the resulting matrix.

(a) **ABC** (b) **ACB** (c) **CAB** (d) **DCF** (e) **BEF** (f) **BFA**
(g) **EBC** (h) **ECB** (i) **EFB** (j) **EFAD**

Question 2 Express each of the following products as a single matrix.

(a) $\begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ (b) $\begin{pmatrix} -2 \\ 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

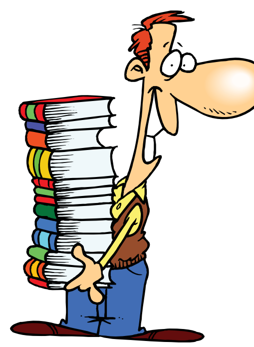
(c) $\begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$



Question 3 Mike is a manager of a building firm.
He is looking after the construction of two types of house, Large and Standard, on three different building sites, site A, site B and site C.
Information on the number of each type of house on each site, the numbers of doors and windows required by each type of house, and the cost of each door and window is shown below.

$$\begin{array}{l} \\ \text{site A} \\ \text{site B} \\ \text{site C} \end{array}\begin{array}{c} \begin{array}{cc} \text{Large} & \text{Standard} \end{array} \\ \begin{pmatrix} 2 & 4 \\ 4 & 1 \\ 5 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{c} \begin{array}{cc} \text{Doors} & \text{Windows} \end{array} \\ \begin{pmatrix} 9 & 8 \\ 5 & 6 \end{pmatrix} \end{array}\begin{array}{l} \\ \text{each Large house} \\ \text{each Standard house} \end{array} \qquad \begin{array}{c} \text{Cost (\$)} \\ \begin{pmatrix} 150 \\ 300 \end{pmatrix} \end{array}\begin{array}{l} \\ \text{each Door} \\ \text{each Window} \end{array}$$

Given $\mathbf{P} = \begin{pmatrix} 2 & 4 \\ 4 & 1 \\ 5 & 0 \end{pmatrix}$ $\mathbf{Q} = \begin{pmatrix} 9 & 8 \\ 5 & 6 \end{pmatrix}$ $\mathbf{R} = \begin{pmatrix} 150 \\ 300 \end{pmatrix}$.



(a) (i) Find the matrix **PQ**.
(ii) What information does this matrix show?

(b) (i) Find the matrix **QR**.
(ii) What information does this matrix show?

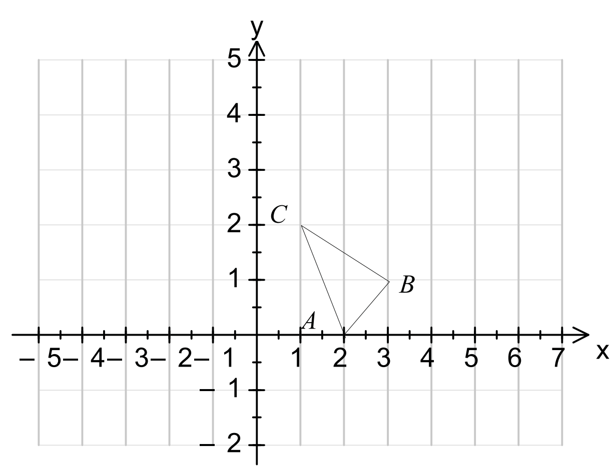
(c) (i) Find the matrix **PQR**.
(ii) What information does this matrix show?

Question 4 The coordinates of triangle *ABC* are shown in the matrix **P**.

$$\mathbf{P} = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} \end{array}$$

$$\mathbf{X} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{Y} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$



(a) Find the matrix **XP**.
(b) Find the matrix **YP**.
(c) Copy the diagram and add the triangles whose coordinates are given by the matrices
(i) **XP**, (ii) **YP**.
(d) What is the relationship between the triangle whose coordinates are given by the matrix **XP** and triangle *ABC*?
(e) What is the relationship between the triangle whose coordinates are given by the matrix **YP** and triangle *ABC*?
(f) Find the matrix **XP**.
(g) Find the matrix **YXP**.
(h) What can you deduce about the triangle whose coordinates are given by the matrix **YXP** and the triangle whose coordinates are given by the matrix **XYP**?

Inverses of Square Matrices of Order 2 (1).

Example 1 Consider the matrices $\mathbf{P} = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}$.

Then $\mathbf{PQ} = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$, the **identity** matrix.

and $\mathbf{QP} = \begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$, the **identity** matrix.

Hence **PQ** = **I** and **QP** = **I**.

> When square matrices **P** and **Q** are such that **PQ** = **I** and **QP** = **I**, where **I** is the identity matrix, **P** is said to be the **inverse** of **Q**, and **Q** is said to be the **inverse** of **P**.
> If **A** and **B** are square matrices such that **AB** = **I**, then it also can be shown that **BA** = **I**.
>
> The inverse of a matrix **R** is denoted by $\mathbf{R}^{-1}$. Thus $\mathbf{R}\,\mathbf{R}^{-1} = \mathbf{I}$ and $\mathbf{R}^{-1}\,\mathbf{R} = \mathbf{I}$.

Question 1 In each part of this question, show that each matrix is the inverse of the other. (i.e. show that **PQ** = **I** and **QP** = **I**.)

(a) $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$ (b) $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

(c) $\mathbf{P} = \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ (d) $\mathbf{P} = \begin{pmatrix} 7 & -5 \\ -4 & 3 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix}$

(e) $\mathbf{P} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$ (f) $\mathbf{P} = \begin{pmatrix} 4 & -9 \\ -3 & 7 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 7 & 9 \\ 3 & 4 \end{pmatrix}$

Question 2 By studying the values of the elements in the pairs of matrices in Question 1, write down the inverses of each of these matrices.
Check each answer by multiplying it by the original matrix.

(a) $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -3 & 2 \\ 4 & -3 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 4 & -5 \\ -7 & 9 \end{pmatrix}$ (g) $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ (h) $\begin{pmatrix} 4 & 7 \\ -3 & -5 \end{pmatrix}$

Question 3 The following matrices are inverses of each other. Find the values of x and y.

(a) $\begin{pmatrix} 11 & 12 \\ x & 11 \end{pmatrix}$ and $\begin{pmatrix} 11 & -12 \\ -10 & y \end{pmatrix}$ (b) $\begin{pmatrix} 16 & 7 \\ 9 & 4 \end{pmatrix}$ and $\begin{pmatrix} 4 & -7 \\ x & y \end{pmatrix}$

(c) $\begin{pmatrix} 11 & x \\ 9 & 5 \end{pmatrix}$ and $\begin{pmatrix} y & -6 \\ -9 & 11 \end{pmatrix}$ (d) $\begin{pmatrix} 7 & 5 \\ y & 3 \end{pmatrix}$ and $\begin{pmatrix} x & -5 \\ -4 & 7 \end{pmatrix}$

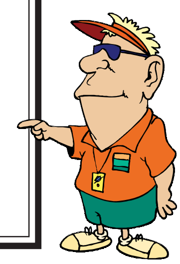
(e) $\begin{pmatrix} 9 & 7 \\ -4 & -3 \end{pmatrix}$ and $\begin{pmatrix} x & y \\ 4 & 9 \end{pmatrix}$ (f) $\begin{pmatrix} x & -5 \\ y & 4 \end{pmatrix}$ and $\begin{pmatrix} 4 & 5 \\ -5 & 6 \end{pmatrix}$

Determinants of Square Matrices of Order 2.

In a square matrix of order 2 the **leading**, or **main**, **diagonal** is the diagonal which goes from top left to bottom right. (The diagonal from $\boldsymbol{a}$ to $\boldsymbol{d}$ in the matrix opposite.) $\begin{pmatrix} \boldsymbol{a} & b \\ c & \boldsymbol{d} \end{pmatrix}$

The **determinant** of a 2 x 2 matrix is defined to be the value of the product of the elements in the leading diagonal minus the value of the product of the elements in the other diagonal.

> When $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant of the matrix **M** is $ad - bc$.
>
> The determinant of the matrix **M** is denoted by det **M** or $|\mathbf{M}|$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.
>
> Thus det $\mathbf{M} = ad - bc$.



Question 1 Calculate the determinant of the following matrices.

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 8 \\ 3 & 6 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 4 \\ 1 & -2 \end{pmatrix}$ (d) $\begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}$ (e) $\begin{pmatrix} 2 & -4 \\ -6 & 0 \end{pmatrix}$ (f) $\begin{pmatrix} 3 & 4 \\ -3 & -2 \end{pmatrix}$

Question 2 Calculate the value(s) of x which makes the determinant of each of the following matrices equal to one.

(a) $\begin{pmatrix} 1 & 2 \\ 3 & x \end{pmatrix}$ (b) $\begin{pmatrix} -2 & 8 \\ x & -3 \end{pmatrix}$ (c) $\begin{pmatrix} x & -4 \\ 3 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 5 & x \\ 3 & 2 \end{pmatrix}$ (e) $\begin{pmatrix} x & 1 \\ 3 & x \end{pmatrix}$ (f) $\begin{pmatrix} x & 2x \\ 3 & 4 \end{pmatrix}$

Question 3 How many times is the value of the determinant of a 2 x 2 matrix increased when each element of the matrix is multiplied by

(a) 2 (b) 3 (c) -2 (d) -3 (e) k ?

Question 4 Given $\mathbf{P} = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$

(a) Find (i) det **P** (ii) det **Q** (iii) **PQ** (iv) **QP** (v) det (**PQ**) (vi) det (**QP**) .
(b) What do you notice about your answers to the determinants in (a) ?

Question 5 Given $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} \frac{1}{2} & -1 \\ -\frac{1}{2} & 2 \end{pmatrix}$

(a) Find (i) det **A** (ii) det **B** (iii) **AB** (iv) **BA** .
(b) What can you say about the matrices **A** and **B** ?
(c) What do you notice about det **A** and det **B** ?

Question 6 Given $\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 4 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 2 \\ -4 & 4 \end{pmatrix}$

(a) Find (i) det **A** (ii) det **B** (iii) **AB** (iv) **BA** .
(b) Note that **AB** and **BA** are both multiples of **I**.
Write down the value of k, where $\mathbf{AB} = k\mathbf{I}$.

Inverses of Square Matrices of Order 2 (2)

Provided det $\mathbf{M} \neq 0$, the matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has an inverse, and $\mathbf{M}^{-1} = \dfrac{1}{\det \mathbf{M}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

When det $\mathbf{M} = 0$ (i.e. $ad - bc = 0$), the matrix $\mathbf{M}$ does not have an inverse.

A matrix which does not have an inverse is called a **singular** matrix.

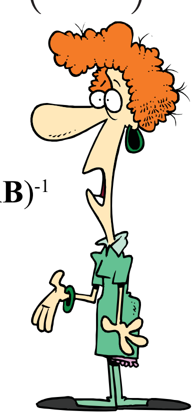
Question 1 State whether each of these matrices has an inverse.
Find the inverse where it exists.

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 8 \\ 3 & 6 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & 3 \\ -2 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$ (e) $\begin{pmatrix} 2 & -4 \\ -6 & 0 \end{pmatrix}$ (f) $\begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}$

(g) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (h) $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$ (i) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (j) $\begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}$ (k) $\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}$ (l) $\begin{pmatrix} 3 & 6 \\ -2 & -4 \end{pmatrix}$

Question 2 Find the values of x for which the following matrices are singular.

(a) $\begin{pmatrix} 1 & 2 \\ 3 & x \end{pmatrix}$ (b) $\begin{pmatrix} -2 & 5 \\ x & -3 \end{pmatrix}$ (c) $\begin{pmatrix} x & -4 \\ 3 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 5 & x \\ 4 & 2 \end{pmatrix}$ (e) $\begin{pmatrix} x & 1 \\ 9 & x \end{pmatrix}$ (f) $\begin{pmatrix} x & 2x \\ 3 & 4 \end{pmatrix}$



Question 3 Given $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$, find

(a) $\mathbf{AB}$ (b) $\mathbf{BA}$ (c) $\mathbf{A}^{-1}$ (d) $\mathbf{B}^{-1}$ (e) $(\mathbf{AB})^{-1}$
(f) $(\mathbf{BA})^{-1}$ (g) $\mathbf{A}^{-1}\,\mathbf{B}^{-1}$ (h) $\mathbf{B}^{-1}\,\mathbf{A}^{-1}$.
Which pairs of answers are equal to each other?

Question 4 Given $\mathbf{P} = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$,

(a) $\mathbf{P}^2 = \begin{pmatrix} 19 & x \\ 5 & 4 \end{pmatrix}$, find x.
(b) Find $\mathbf{P}^{-1}$.
(c) Find the inverse of $\mathbf{P}^2$ (i.e. $(\mathbf{P}^2)^{-1}$).
(d) Find the square of the inverse of $\mathbf{P}$ (i.e. $(\mathbf{P}^{-1})^2$).
(e) What do you notice about your answers to (c) and (d) ?

Question 5 Given that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, find the values of a, b, c and d.

Question 6 Given $\mathbf{Q} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$,

(a) find the two values of h which makes $\mathbf{Q} - h\mathbf{I}$ singular,
(b) find the two values of k which makes $\mathbf{Q}^{-1} - k\mathbf{I}$ singular.

Solving Equations with Matrices.

When **A** and **B** are given 2 x 2 matrices, the equation **AX** = **B** can be solved, to find the 2 x 2 matrix **X**, provided det **A** $\neq$ 0.

This is done by **pre**-multiplying each side of the equation by $\mathbf{A}^{-1}$ (which exists, since det **A** $\neq$ 0).

$$\begin{aligned} &\text{Thus} \quad \mathbf{AX} = \mathbf{B} \\ \Rightarrow \quad &\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B} \\ \Rightarrow \quad &\mathbf{IX} = \mathbf{A}^{-1}\mathbf{B} \qquad (\text{since } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}) \\ \Rightarrow \quad &\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} \qquad (\text{since } \mathbf{IX} = \mathbf{X}) \end{aligned}$$

In the same way the equation **YA** = **B** can be solved, provided det **A** $\neq$ 0, by **post**-multiplying each side of the equation by $\mathbf{A}^{-1}$.

$$\begin{aligned} &\text{Thus} \quad \mathbf{YA} = \mathbf{B} \\ \Rightarrow \quad &\mathbf{YAA}^{-1} = \mathbf{BA}^{-1} \\ \Rightarrow \quad &\mathbf{YI} = \mathbf{BA}^{-1} \qquad (\text{since } \mathbf{AA}^{-1} = \mathbf{I}) \\ \Rightarrow \quad &\mathbf{Y} = \mathbf{BA}^{-1} \qquad (\text{since } \mathbf{YI} = \mathbf{Y}) \end{aligned}$$

Question 1 Solve the following matrix equations.

(a) $\begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$ (b) $\mathbf{X}\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 3 & 1 \\ 0 & -1 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 0 & -3 \\ 2 & 5 \end{pmatrix}$ (e) $\mathbf{X}\begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ 0 & 9 \end{pmatrix}$ (f) $\mathbf{X}\begin{pmatrix} 2 & 5 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 6 & -4 \\ 2 & 8 \end{pmatrix}$

Inverse matrices can be used, in a similar way, to solve simultaneous equations.

Example 1 Solve the simultaneous equations

$$4x + 3y = 5$$
$$x + 2y = -10$$

These equations can be written as $\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -10 \end{pmatrix}$.

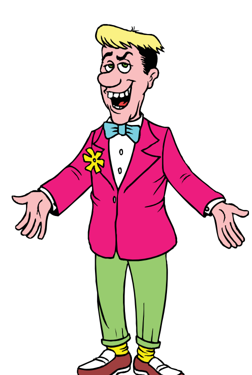
Pre-multiply each side of the equation by the inverse of $\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$ which is $\frac{1}{5}\begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}$.

$$\text{Thus} \quad \frac{1}{5}\begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ -10 \end{pmatrix}$$

$$\Rightarrow \quad \frac{1}{5}\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 40 \\ -45 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ -9 \end{pmatrix}$$

So the solution is $x = 8$, $y = -9$.



Question 2 Solve the equations

(a) $\begin{pmatrix} 8 & 3 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 10 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 12 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 17 \end{pmatrix}$

Miscellaneous Matrix Questions (1).

Question 1 Express each of the following as a single matrix.

(a) $\begin{pmatrix}5\\4\end{pmatrix}-\begin{pmatrix}2\\-3\end{pmatrix}$ (b) $\begin{pmatrix}3 & -5\end{pmatrix}+2\begin{pmatrix}-3 & 3\end{pmatrix}$ (c) $3\begin{pmatrix}-1\\2\end{pmatrix}+\begin{pmatrix}5\\-4\end{pmatrix}$

(d) $3\begin{pmatrix}5\\2\end{pmatrix}+2\begin{pmatrix}3\\-5\end{pmatrix}$ (e) $2\begin{pmatrix}5\\2\end{pmatrix}+3\begin{pmatrix}3\\-5\end{pmatrix}+\begin{pmatrix}-6\\4\end{pmatrix}$ (f) $3\begin{pmatrix}-3\\2\end{pmatrix}+2\begin{pmatrix}2\\-5\end{pmatrix}-\begin{pmatrix}-2\\2\end{pmatrix}$

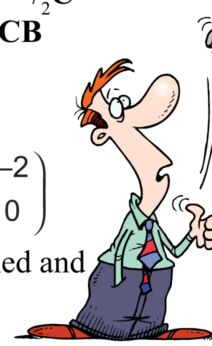
(g) $3\begin{pmatrix}0 & 1\\-3 & 2\end{pmatrix}+2\begin{pmatrix}1 & 0\\-2 & 3\end{pmatrix}-\begin{pmatrix}5 & 4\\-1 & 3\end{pmatrix}$ (h) $\begin{pmatrix}0 & 1\\-2 & 3\end{pmatrix}-2\begin{pmatrix}1 & 0\\-3 & 2\end{pmatrix}-3\begin{pmatrix}5 & 4\\-1 & 3\end{pmatrix}$

(i) $\begin{pmatrix}1 & -2\\0 & 4\end{pmatrix}\begin{pmatrix}2 & -3\\0 & 1\end{pmatrix}$ (j) $\begin{pmatrix}2 & -3\\0 & 1\end{pmatrix}\begin{pmatrix}4 & -6\\0 & -2\end{pmatrix}$ (k) $\begin{pmatrix}0 & 1\\2 & 3\end{pmatrix}\begin{pmatrix}-1 & 0\\1 & -2\end{pmatrix}\begin{pmatrix}3 & -2\\0 & 1\end{pmatrix}$

Question 2 Given that $\mathbf{A}=\begin{pmatrix}0 & 3\\5 & -1\end{pmatrix}$ $\mathbf{B}=\begin{pmatrix}1 & -2\\0 & 4\end{pmatrix}$ $\mathbf{C}=\begin{pmatrix}4 & -6\\-2 & 0\end{pmatrix}$,

express each of the following as a single matrix.

(a)	2**A**	(b)	$^1/_2$**C**	(c)	- **B**	(d)	**A** - 2**B**
(e)	**B** - **C**	(f)	**A** + **B** + **C**	(g)	2**A** + 3**B**	(h)	**B** - $^1/_2$**C**
(i)	**A** + 2**B** - 3**C**	(j)	**AB**	(k)	**ABC**	(l)	**ACB**



Question 3 Given that

$\mathbf{A}=\begin{pmatrix}4\\2\end{pmatrix}$ $\mathbf{B}=\begin{pmatrix}3 & 6\end{pmatrix}$ $\mathbf{C}=\begin{pmatrix}2 & -3\\0 & 1\end{pmatrix}$ $\mathbf{D}=\begin{pmatrix}3 & 0\\1 & -2\end{pmatrix}$ $\mathbf{E}=\begin{pmatrix}6 & -4\end{pmatrix}$ $\mathbf{F}=\begin{pmatrix}-2\\0\end{pmatrix}$

say whether or not each of the following combinations of matrices can be formed and evaluate those which can.

(a)	**A** + **B**	(b)	**B** + **C**	(c)	**C** - **D**	(d)	2**A** + **F**	(e)	2**B** + **C**
(f)	2**B** - 3**E**	(g)	$^1/_3$**B** + **E**	(h)	**C** - **D** + **F**	(i)	**D** + 2**E**	(j)	2**A** + **D**
(k)	$^1/_2$**F** - **A**	(l)	2**B** - 2**E** + **D**	(m)	$^1/_2$**C** + **D**				

Question 4 Given that

$\mathbf{A}=\begin{pmatrix}4\\2\end{pmatrix}$ $\mathbf{B}=\begin{pmatrix}3 & 6\end{pmatrix}$ $\mathbf{C}=\begin{pmatrix}2 & -3\\0 & 1\end{pmatrix}$ $\mathbf{D}=\begin{pmatrix}3 & 0\\1 & -2\end{pmatrix}$ $\mathbf{E}=\begin{pmatrix}6 & -4\end{pmatrix}$ $\mathbf{F}=\begin{pmatrix}-2\\0\end{pmatrix}$

perform, where possible, the following products. If any are impossible, then say so.

(a)	**AB**	(b)	**BA**	(c)	**AC**	(d)	**CA**	(e)	**AF**
(f)	**FA**	(g)	**BC**	(h)	**CB**	(i)	**BE**	(j)	**EB**
(k)	**CD**	(l)	**DC**						

Question 5 Find (i) the determinant,
(ii) the inverse

of each of the following matrices.

(a) $\begin{pmatrix}7 & 3\\9 & 4\end{pmatrix}$ (b) $\begin{pmatrix}5 & 3\\6 & 4\end{pmatrix}$ (c) $\begin{pmatrix}-3 & -5\\4 & 7\end{pmatrix}$ (d) $\begin{pmatrix}-3 & 6\\-2 & 5\end{pmatrix}$

Question 6 Find p, q, r and s from $\begin{pmatrix}1 & -2\\ p & 3\end{pmatrix}\begin{pmatrix}q & 1\\ 4 & r\end{pmatrix}=\begin{pmatrix}-6 & 3\\ s & 2\end{pmatrix}$

Question 7 Find the values of x and y in the following.

(a) $\begin{pmatrix}2x\\ y\end{pmatrix}=\begin{pmatrix}-6\\ 0\end{pmatrix}$ (b) $2\begin{pmatrix}3x\\ y\end{pmatrix}=\begin{pmatrix}12\\ -1\end{pmatrix}$ (c) $3\begin{pmatrix}x\\ 2\end{pmatrix}=\begin{pmatrix}-6\\ y\end{pmatrix}$ (d) $2\begin{pmatrix}x & 3\\ -2 & y\end{pmatrix}=\begin{pmatrix}4 & 6\\ -4 & -6\end{pmatrix}$

(e) $2\begin{pmatrix}x & 0\\ 6 & -y\end{pmatrix}=3\begin{pmatrix}y & 0\\ 4 & 2\end{pmatrix}$ (f) $x\begin{pmatrix}4\\ 1\end{pmatrix}-y\begin{pmatrix}3\\ 2\end{pmatrix}=\begin{pmatrix}5\\ -10\end{pmatrix}$ (g) $\begin{pmatrix}x & 0\\ 1 & y\end{pmatrix}\begin{pmatrix}2\\ 3\end{pmatrix}=\begin{pmatrix}6\\ -1\end{pmatrix}$

(h) $\begin{pmatrix}13 & x\\ 9 & 7\end{pmatrix}\begin{pmatrix}y & -10\\ -9 & 13\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ (i) $\begin{pmatrix}x & 2\\ y & -3\end{pmatrix}\begin{pmatrix}6 & 4\\ 5 & 3\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

(j) $\begin{pmatrix}3 & 1\\ 2 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}9\\ 1\end{pmatrix}$ (k) $\begin{pmatrix}x & y\\ y & x\end{pmatrix}\begin{pmatrix}3\\ 1\end{pmatrix}=\begin{pmatrix}-5\\ 1\end{pmatrix}$ (l) $\begin{pmatrix}3 & x\\ -2 & 1\end{pmatrix}\begin{pmatrix}y\\ 5\end{pmatrix}=\begin{pmatrix}-2\\ 3\end{pmatrix}$

Question 8 The following simultaneous equations can be written in the form $\mathbf{A}\begin{pmatrix}x\\ y\end{pmatrix}=\mathbf{B}$.

Find, in each case, the matrices **A** and **B**.

(a) $3x-2y=4$, $2x+5y=9$ (b) $3x+2y+7=0$, $4x+y-4=0$ (c) $2x - y = 10$, $5y = 10 + 6x$

(d) $2x=25-3y$, $3y=x-17$ (e) $x+2y=12$, $3y-2x=25$ (f) $2x+y+12=0$, $3y-4x=34$

Question 9 Find the matrix **X** in the following equations.

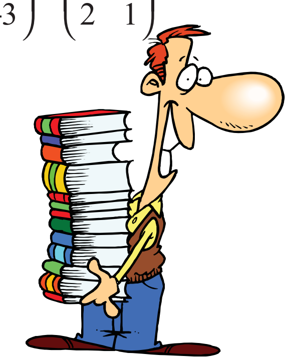
(a) $\mathbf{X}+2\begin{pmatrix}5\\ -1\end{pmatrix}=\begin{pmatrix}9\\ 4\end{pmatrix}$ (b) $2\mathbf{X}-\begin{pmatrix}2\\ 4\end{pmatrix}=\begin{pmatrix}-6\\ 4\end{pmatrix}$ (c) $3\mathbf{X}-\begin{pmatrix}-8\\ 16\end{pmatrix}=5\mathbf{X}$

(d) $\mathbf{X}-\begin{pmatrix}-4\\ 12\end{pmatrix}=5\mathbf{X}$ (e) $5\mathbf{X}=\begin{pmatrix}25 & -5\\ 15 & 10\end{pmatrix}$ (f) $2\mathbf{X}+\begin{pmatrix}1 & -1\\ 2 & 0\end{pmatrix}=\begin{pmatrix}7 & 3\\ 0 & 6\end{pmatrix}$

(g) $2\mathbf{X}=\begin{pmatrix}1 & 1\\ -2 & -4\end{pmatrix}\begin{pmatrix}2 & 1\\ 0 & -1\end{pmatrix}$ (h) $2\mathbf{X}+\begin{pmatrix}2 & 3\\ 3 & -2\end{pmatrix}=\begin{pmatrix}2 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & -1\\ 2 & 3\end{pmatrix}$

(i) $\begin{pmatrix}3 & 1\\ 2 & 1\end{pmatrix}\mathbf{X}=\begin{pmatrix}2 & 4\\ -1 & -3\end{pmatrix}$ (j) $\begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}\mathbf{X}=\begin{pmatrix}10 & 10\\ 5 & 10\end{pmatrix}$ (k) $\mathbf{X}\begin{pmatrix}2 & 4\\ -1 & -3\end{pmatrix}=\begin{pmatrix}3 & 1\\ 2 & 1\end{pmatrix}$

(l) $\mathbf{X}\begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix}=\begin{pmatrix}19 & 13\\ 2 & 4\end{pmatrix}$ (m) $\mathbf{X}+\mathbf{X}\begin{pmatrix}3 & 2\\ 1 & 0\end{pmatrix}=\begin{pmatrix}-2 & 0\\ 6 & 4\end{pmatrix}$



Question 10 Given $\mathbf{A}=\begin{pmatrix}4 & 3\\ 0 & 2\end{pmatrix}$ $\mathbf{B}=\begin{pmatrix}\frac{1}{4} & x\\ 0 & \frac{1}{2}\end{pmatrix}$ $\mathbf{C}=\begin{pmatrix}12 & 4\\ -8 & y\end{pmatrix}$,

(a) Evaluate $\mathbf{A}^2$.
(b) Find the value of x which makes **AB** the identity matrix.
(c) Find the value of y which makes the determinant of **A** equal to the determinant of **C**.
(d) Find the determinant of **AC**.

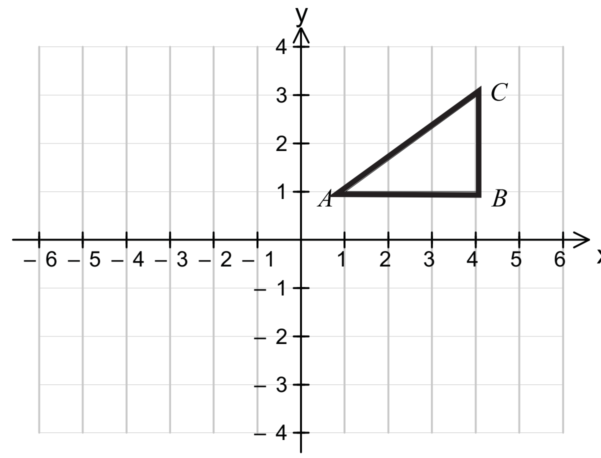
Question 11 Given $\mathbf{A}=\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}$ and $\mathbf{A}^2=p\mathbf{A}+q\mathbf{I}$, find the values of p and q.

Miscellaneous Matrix Questions (2).

Question 1 The coordinates of triangle *ABC* are shown in the matrix **P**.

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \mathbf{P} = & \begin{pmatrix} 1 & 4 & 4 \\ 1 & 1 & 3 \end{pmatrix} \end{array}$$

$$\mathbf{X} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



(a) Find the matrix **XP**.
(b) Find the matrix $\mathbf{X}^{-1}\mathbf{P}$.
(c) Find the matrix $\mathbf{X}^{-1}\mathbf{XP}$.
(d) Find the matrix $\mathbf{X}^2\mathbf{P}$.
(e) Copy the diagram and add the triangles whose coordinates are given by the matrices
  (i) **XP**,
  (ii) $\mathbf{X}^{-1}\mathbf{P}$,
  (iii) $\mathbf{X}^{-1}\mathbf{XP}$,
  (iv) $\mathbf{X}^2\mathbf{P}$.
(f) What is the relationship between the triangle whose coordinates are given by the matrix **XP** and triangle *ABC*?
(g) What is the relationship between the triangle whose coordinates are given by the matrix $\mathbf{X}^{-1}\mathbf{P}$ and triangle *ABC*?
(h) What is the relationship between the triangle whose coordinates are given by the matrix $\mathbf{X}^{-1}\mathbf{XP}$ and triangle *ABC*?
(i) What is the relationship between the triangle whose coordinates are given by the matrix $\mathbf{X}^2\mathbf{P}$ and triangle *ABC*?

Question 2 The marks scored by Ann, Brian and Chris in Maths and Science exams are given in the matrix below.

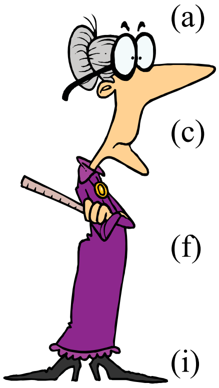
$$\begin{array}{rc} & \begin{array}{ccc} \text{Ann} & \text{Brian} & \text{Chris} \end{array} \\ \begin{array}{r} \text{Maths} \\ \text{Science} \end{array} & \begin{pmatrix} 82 & 70 & 76 \\ 64 & 72 & 68 \end{pmatrix} \end{array}$$

Given $\mathbf{P} = \begin{pmatrix} 82 & 70 & 76 \\ 64 & 72 & 68 \end{pmatrix}$, $\mathbf{Q} = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{R} = \begin{pmatrix} 1 & 1 \end{pmatrix}$.

(a) Find **PQ**.
What information does this matrix show?

(b) Find **RP**.
What information does this matrix show?

Question 3 Find the value of each letter in these matrix equations.



(a) $\begin{pmatrix} 30 \\ u \end{pmatrix} + 2\begin{pmatrix} -6 \\ 5 \end{pmatrix} = 3\begin{pmatrix} v \\ -2 \end{pmatrix}$ (b) $3\begin{pmatrix} 1 & 1 \\ 0 & x \end{pmatrix} + w\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} y & 5 \\ z & 0 \end{pmatrix}$

(c) $\begin{pmatrix} a & 4 \\ 0 & b \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 14 \\ 10 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & c \\ -1 & d \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$ (e) $\begin{pmatrix} -1 & 1 \\ e & -1 \end{pmatrix}\begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} f \\ 7 \end{pmatrix}$

(f) $\begin{pmatrix} 5 & 6 \\ 2 & g \end{pmatrix}\begin{pmatrix} h \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$ (g) $\begin{pmatrix} 2j & 1 \\ 3 & j \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ k \end{pmatrix}$ (h) $\begin{pmatrix} 4 & 2 \end{pmatrix}\begin{pmatrix} m & 2 \\ -1 & n \end{pmatrix} = \begin{pmatrix} 2 & 6 \end{pmatrix}$

(i) $\begin{pmatrix} 3 & p \\ q & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & r \end{pmatrix} = \begin{pmatrix} 9 & 16 \\ 7 & s \end{pmatrix}$ (j) $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 3 & y \\ x & 1 \end{pmatrix} = \begin{pmatrix} t & 4 \\ 29 & z \end{pmatrix}$

Question 4 Given $\mathbf{B} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

(a) show that $\mathbf{B} = 2\mathbf{I}$,
(b) hence write down (i) $\mathbf{B}^5$, (ii) $\mathbf{B}^{-1}$, (iii) $\mathbf{B}^{-5}$.

Question 5 Given $\mathbf{X} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$, $\mathbf{Y} = \begin{pmatrix} 1 & 4 \\ 3 & -5 \end{pmatrix}$, $\mathbf{Z} = \begin{pmatrix} 4 & 0 \\ -3 & 3 \end{pmatrix}$.

(a) Find (i) **XY**, (ii) **XZ**.
(b) When x, y and z are numbers and $xy = xz$, what can be deduced? Is a similar result true for matrices?

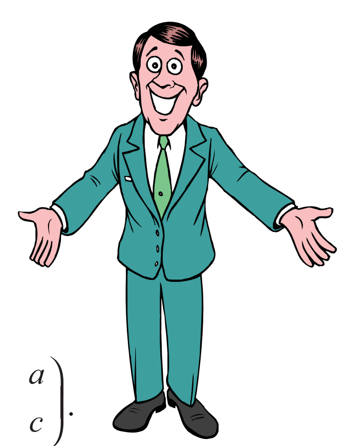
Question 6 (a) Find $\mathbf{M}\begin{pmatrix} 1 \\ 5 \end{pmatrix}$, where **M** is the inverse of the matrix $\begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix}$.

(b) Hence write down the solutions to the simultaneous equations

$$x + 4y = 1,$$
$$2x + 5y = 5.$$

Question 7 Given $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$.

(a) Find (i) **AB**, (ii) **BA.**
(b) Find the matrix **X** such that **XAB** = **BA**.
(c) Find the matrix **Y** such that $\mathbf{AY} = \mathbf{B}^2$.
(d) Find the matrix **Z** such that $\mathbf{AZ} = \mathbf{B}^{-1}$.



Question 8 Given $\mathbf{A} = \begin{pmatrix} 5 & -3 \\ 4 & -2 \end{pmatrix}$,

(a) find the two values of x which makes $\mathbf{A} - x\mathbf{I}$ singular,
(b) find the two values of y which makes $\mathbf{A}^{-1} - y\mathbf{I}$ singular.

Question 9 If $ad \neq bc$, find the matrix **X** such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathbf{X} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$.

Matrices and Transformations (Introduction).

A geometrical transformation is a process by which an **image** point is obtained from an **object** point.

An object point P, with coordinates $(a\ ,\ b)$, is represented by the column matrix $\begin{pmatrix} a \\ b \end{pmatrix}$.

This column matrix is pre-multiplied by a 2 x 2 matrix to give a column matrix, $\begin{pmatrix} a' \\ b' \end{pmatrix}$, which

represents the coordinates (a', b') of the image point.

Thus, with an object point (6 , 7) and a matrix $\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$ we get $\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 7 \end{pmatrix} = \begin{pmatrix} -8 \\ 19 \end{pmatrix}$.

Hence the image point $(-8, 19)$ is obtained from the object point (6 , 7) by the transformation represented by the matrix .

Example 1

The triangle with vertices A (2 , 1), B (4 , 1) and C (4 , 2) is transformed by the transformation represented by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

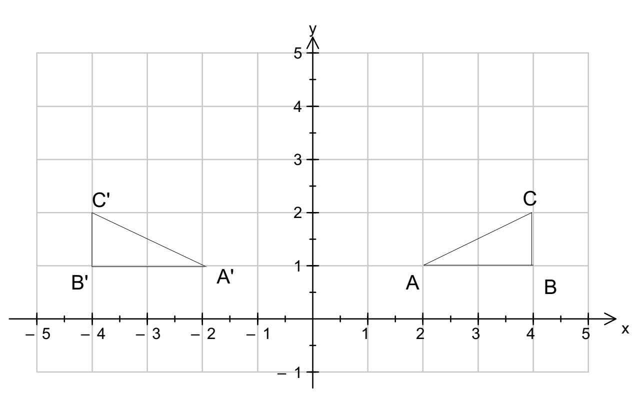
(a) Find the coordinates of the vertices of the image triangle.
(b) Draw the object and image.
(c) Describe the transformation represented by the matrix.

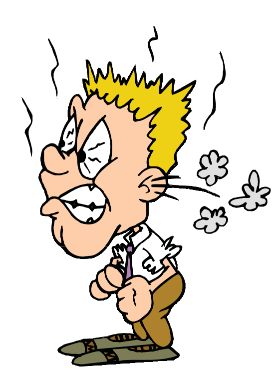
Answer.

(a)

Object point	Calculation	Image point
A (2 , 1)	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$	A' (−2 , 1)
B (4 , 1)	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$	B' (−4 , 1)
C (4 , 2)	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$	C' (−4 , 2)

(b)





(c) From the diagram it is clear that the image is obtained from the object by a reflection in the y-axis.

The matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is said to represent a *reflection in the y-axis*.

The mirror line is the y-axis. The coordinates of points which lie on the y-axis do not change under this transformation. Such points are said to be **invariant**.

Question 1 (a) Copy and complete the tables below.

(b) Draw and label x and y axes from 0 to 8.
On your diagram draw the two objects and their images.

(c) Describe **fully** the transformation represented by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

(d) Find the area scale factor for each shape.
Compare your answers with the determinant of the matrix.

First shape

Object point	Calculation	Image point
$A\ (2\ ,\ 1)$	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \ \\ \ \end{pmatrix}$	$A'\ (\quad ,\quad)$
$B\ (4\ ,\ 1)$	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} \ \\ \ \end{pmatrix}$	$B'\ (\quad ,\quad)$
$C\ (4\ ,\ 2)$	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} \ \\ \ \end{pmatrix}$	$C'\ (\quad ,\quad)$

Second shape

Object point	Calculation	Image point
$O\ (0\ ,\ 0)$	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \ \\ \ \end{pmatrix}$	$O'\ (\quad ,\quad)$
$P\ (1\ ,\ 0)$	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \ \\ \ \end{pmatrix}$	$P'\ (\quad ,\quad)$
$Q\ (1\ ,\ 1)$	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \ \\ \ \end{pmatrix}$	$Q'\ (\quad ,\quad)$
$R\ (0\ ,\ 1)$	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \ \\ \ \end{pmatrix}$	$R'\ (\quad ,\quad)$

Question 2 Repeat Question 1 with the matrix $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

Use the same object shapes, but draw axes from 0 to 12.

Question 3 Repeat Question 1 with the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Use the same object shapes, but draw axes from -4 to 4.

Question 4 Repeat Question 1 with the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Use the same object shapes, but draw axes from -4 to 4.

Single Matrix Transformations.

When calculating the coordinates of the image shapes it is more efficient to combine the column matrices into a single matrix and pre-multiply this matrix by the transformation matrix.
Thus, in Example 1 of the Introduction, we could perform the calculation

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 2 & 4 & 4 \\ 1 & 1 & 2 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} A' & B' & C' \end{array} \\ \begin{pmatrix} -2 & -4 & -4 \\ 1 & 1 & 2 \end{pmatrix} \end{array}.$$

Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, the Origin is always transformed to itself.

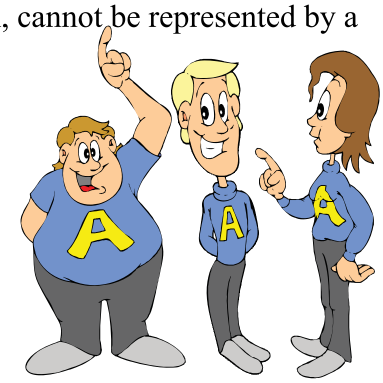
The Origin is **invariant** under a matrix transformation.

The only transformations which can be represented by 2 x 2 matrices are those which have the Origin as an invariant point.

Transformations with a single invariant point, such as a rotation or an enlargement, can only be represented by a 2 x 2 matrix if this invariant point is the Origin.

Transformations with an invariant line, such as a reflection, one-way stretch or shear, can only be represented by a 2 x 2 matrix if this invariant line passes through the Origin.

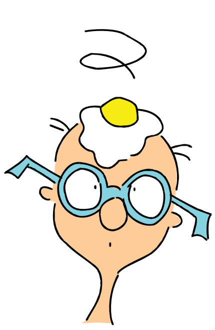
Transformations which have no invariant points, such as a translation, cannot be represented by a 2 x 2 matrix.



Question 1 For each matrix below, with a

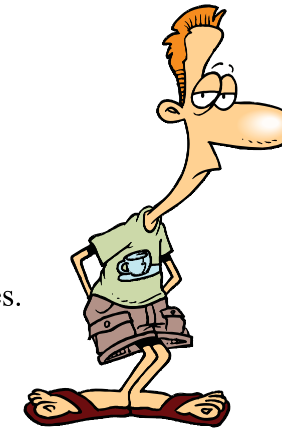
First shape $A\,(2, 1)$ $B\,(3, 1)$ $C\,(3, 3)$
Second shape $O\,(0, 0)$ $P\,(1, 0)$ $Q\,(1, 1)$ $R\,(0, 1)$.

(i) Calculate the coordinates of the two image shapes.
(ii) Draw and label x and y axes from -4 to 6.
On your diagram draw the two object shapes and their images.
(iii) Describe **fully** the **single** transformation which is represented by the matrix.
(iv) Write down the equation of the line whose points are invariant.
(v) Find the area factor for each shape.
Compare your answer with the determinant of the matrix.



(a) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ (g) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ (h) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$



Question 2 For each matrix below, with a

First shape $A\,(2\,,1)$ $B\,(3\,,1)$ $C\,(3\,,3)$

Second shape $O\,(0\,,0)$ $P\,(1\,,0)$ $Q\,(1\,,1)$ $R\,(0\,,1)$.

(i) Calculate the coordinates of the two image shapes.

(ii) Draw and label x and y axes from -6 to 9.
On your diagram draw the two object shapes and their images.

(iii) Describe the **type** of transformation which has taken place.

(iv) Find the area factor for each shape.
Compare your answer with the determinant of the matrix.

(a) $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

(v) Write down the scale factor and centre of this transformation.

(b) $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

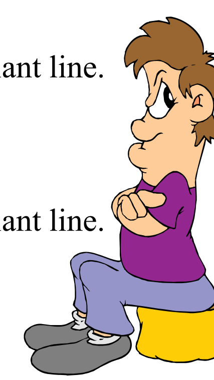
(v) Write down the scale factor and centre of this transformation.

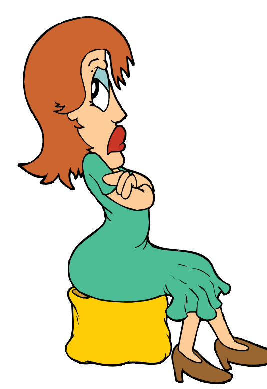
(c) $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

(v) In which direction have the points moved?
Write down the equation of the invariant line.

(d) $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

(v) Write down the equation of the invariant line.





(e) $\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$

(v) Write down the equation of the invariant line.

Question 3 For the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$,

First shape $A\,(2\,,1)$ $B\,(3\,,1)$ $C\,(3\,,3)$

Second shape $O\,(0\,,0)$ $P\,(1\,,0)$ $Q\,(1\,,1)$ $R\,(0\,,1)$.

(i) Calculate the coordinates of the two image shapes.

(ii) Draw and label x and y axes from -1 to 4.
On your diagram draw the two object shapes and their images.

(iii) Describe the **type** of transformation which has taken place.

(iv) Draw the invariant line on your diagram and write down its equation.

(v) In which direction have all the points moved?

Question 4 Repeat Question 3 with the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$.

Draw axes with x values from -3 to 3 and y values from 0 to 9.

Multiple Matrix Transformations (1).

A combination of transformations can be represented by the matrix that is equal to the product of the matrices which represent the individual transformations.
The order in which the product of the matrices is performed is vital.

> If the **first** transformation to be performed is represented by the matrix **M**, and the **second** transformation to be performed is represented by the matrix **N**, then the matrix which represents the combined transformations is represented by the product **NM**.

Example 1 Triangle ABC has vertices $A\ (2\ ,\ 1)$, $B\ (4\ ,\ 1)$ and $C\ (4\ ,\ 2)$.

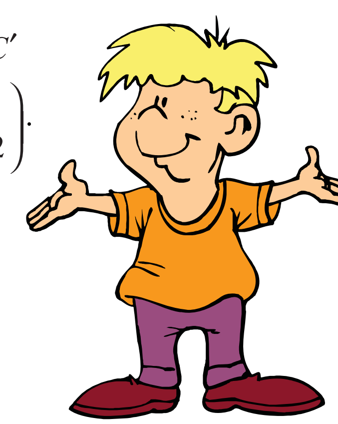
The transformation **T**, reflect in the x-axis, is represented by the matrix **M**, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The transformation **R**, rotate through 90° anticlockwise about (0 , 0), is represented by the matrix **N**, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(a) Transformation **T** maps triangle ABC onto triangle $A'B'C'$.
Calculate the coordinates of triangle $A'B'C'$.

(b) Transformation **R** maps triangle $A'B'C'$ onto triangle $A''B''C'$.
Calculate the coordinates of triangle $A''B''C''$.

(c) Describe the **single** transformation which is equivalent to this combination of these two transformations.



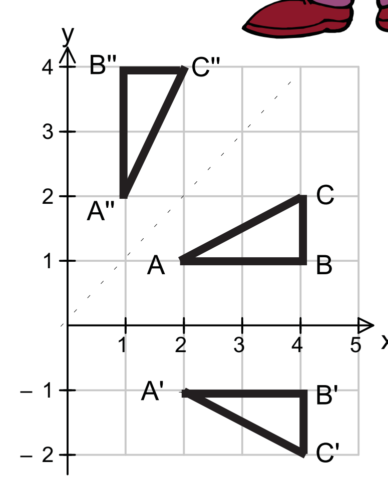
(a) The coordinates are given by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 2 & 4 & 4 \\ 1 & 1 & 2 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} A' & B' & C' \end{array} \\ \begin{pmatrix} 2 & 4 & 4 \\ -1 & -1 & -2 \end{pmatrix} \end{array}.$$

(b) The coordinates are given by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{array}{c} \begin{array}{ccc} A' & B' & C' \end{array} \\ \begin{pmatrix} 2 & 4 & 4 \\ -1 & -1 & -2 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} A'' & B'' & C'' \end{array} \\ \begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & 4 \end{pmatrix} \end{array}.$$

(c) Plotting these points gives this diagram, from which we see that the **single** equivalent transformation is a reflection in the line $y = x$.



Note that the calculations in (a) and (b) are equivalent to

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 4 & 4 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}\begin{pmatrix} 2 & 4 & 4 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & 4 \end{pmatrix}$$

i.e. We can find the matrix which represents the combined transformations directly.

Question 1 Use the transformations and triangle *ABC* given in Example 1.

(a) The matrix **N** transforms triangle *ABC* onto triangle *PQR*.
Calculate the coordinates of triangle *PQR*.

(b) The matrix **M** transforms triangle *PQR* onto triangle *XYZ*.
Calculate the coordinates of triangle *XYZ*.

(c) Calculate the matrix which will map triangle *ABC* directly onto triangle *XYZ*.

(d) Describe **fully** the **single** transformation which is equivalent to this combination of these transformations.

Question 2 Use axes with $-4 \le x \le 4$, $-4 \le y \le 3$.

(a) A triangle has vertices *A* (2 , 1), *B* (4 , 1) and *C* (4 , 2).

It is transformed using the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ onto triangle *A'B'C'*.

Calculate the coordinates of the vertices of triangle *A'B'C'*.
Draw triangle *ABC* and triangle *A'B'C'* on your diagram.

(b) Triangle *A'B'C'* is now transformed onto triangle *A''B''C''* by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Calculate the coordinates of the vertices of triangle *A''B''C''*.
Draw triangle *A''B''C''* on your diagram.

(c) Describe, **fully**, each of these two transformations.

(d) Calculate the matrix which will map triangle *ABC* directly onto triangle *A''B''C''*.

Question 3 Use axes with $-3 \le x \le 3$, $0 \le y \le 2$.

(a) A rectangle has vertices *A* (1 , 0), *B* (3 , 0), *C* (3 , 1) and *D* (1 , 1).

It is transformed using the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ onto *A'B'C'D'*.

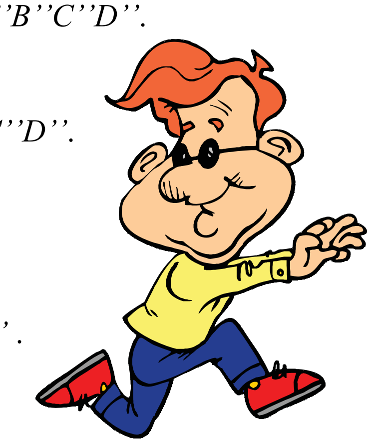
Calculate the coordinates of the vertices of *A'B'C'D'*.
Draw the quadrilaterals *ABCD* and *A'B'C'D'* on your diagram.

(b) *A'B'C'D'* is now transformed onto *A''B''C''D''* by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

Calculate the coordinates of the vertices of the quadrilateral *A''B''C''D''*.
Draw quadrilateral *A''B''C''D''* on your diagram.

(c) Describe, **fully**, each of these two transformations.

(d) Calculate the matrix which maps *ABCD* directly onto *A''B''C''D''*.

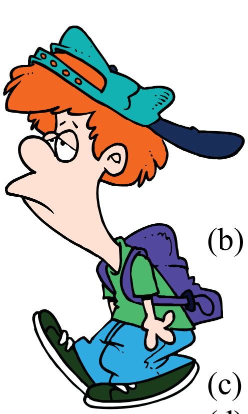


Question 4 Use axes with $-4 \le x \le 4$, $0 \le y \le 4$.

(a) A triangle has vertices *A* (2 , 1), *B* (4 , 1) and *C* (4 , 2).

It is transformed using the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ onto triangle *A'B'C'*.

Calculate the coordinates of the vertices of triangle *A'B'C'*.
Draw triangles *ABC* and *A'B'C'* on your diagram.



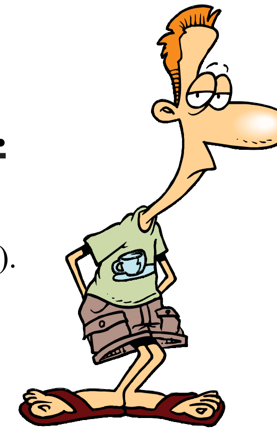
(b) Triangle *A'B'C'* is now transformed onto triangle *A''B''C''* by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Calculate the coordinates of the vertices of triangle *A''B''C''*.
Draw triangle *A''B''C''* on your diagram.

(c) Describe, **fully**, each of these two transformations.

(d) Calculate the matrix which will map triangle *ABC* directly onto triangle *A''B''C''*.
Describe fully the transformation which this matrix represents.

Multiple Matrix Transformations (2).



Question 1 Use axes with $-6 \le x \le 3$, $-3 \le y \le 3$.

(a) A rectangle has vertices A (1 , 0), B (3 , 0), C (3 , 1) and D (1 , 1).

It is transformed using the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ onto $A'B'C'D'$.

Calculate the coordinates of the vertices of $A'B'C'D'$.
Draw the quadrilaterals $ABCD$ and $A'B'C'D'$ on your diagram.

(b) $A'B'C'D'$ is now transformed onto $A''B''C''D''$ by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Calculate the coordinates of the vertices of the quadrilateral $A''B''C''D''$.
Draw quadrilateral $A''B''C''D''$ on your diagram.

(c) $A''B''C''D''$ is now transformed onto $A'''B'''C'''D'''$ by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

Calculate the coordinates of the vertices of the quadrilateral $A'''B'''C'''D'''$.
Draw quadrilateral $A'''B'''C'''D'''$ on your diagram.

(d) Describe, fully, each of these three transformations.

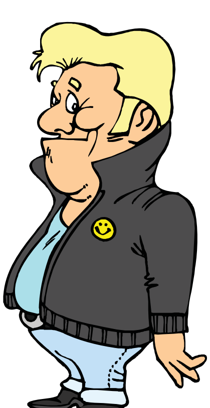
(e) Calculate the matrix which maps $ABCD$ directly onto $A'''B'''C'''D'''$.
Describe, fully, the transformation represented by this matrix.

Question 2 Use axes with $-3 \le x \le 3$, $-3 \le y \le 2$.

The transformation P is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The transformation Q is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Triangle T has vertices (1 , 1), (3 , 1) and (3 , 2).



(a) The transformation P maps triangle T onto triangle T_1, so that $P(T) = T_1$.
Calculate the coordinates of the vertices of triangle T_1.
Draw and label triangles T and T_1.

(b) The transformation Q maps triangle T_1 onto triangle T_2, so that $Q(T_1) = T_2$.
Calculate the coordinates of the vertices of triangle T_2.
Draw and label triangle T_2.

(c) Given that $PQ(T) = T_3$, calculate the coordinates of the vertices of triangle T_3.
Draw and label triangle T_3.

(d) The transformation R maps triangle T_2 onto triangle T_3, so that $R(T_2) = T_3$.
Describe, fully, transformation R.
Write down the matrix which represents transformation R.

Question 3 The transformation "reflection in the y-axis" is followed by a second transformation. The combined effect of these two transformations is the transformation which is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Describe the second transformation and write down the matrix which represents it.

Question 4 A certain transformation is followed by the transformation "rotation through 90°, anticlockwise, about (0 , 0)".
The combined effect of these two transformations is the transformation which is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
Describe the first transformation and write down the matrix which represents it.

Question 5 The transformation "enlargement, scale factor 2, centre (0 , 0)" is followed by a second transformation.
The combined effect of these two transformations is the transformation which is represented by the matrix $\begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$.
Describe the second transformation and write down the matrix which represents it.

Question 6 A certain transformation is followed by the transformation "one-way stretch, scale factor 2, with the y-axis invariant".
The combined effect of these two transformations is the transformation which is represented by the matrix $\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$.
Describe the first transformation and write down the matrix which represents it.

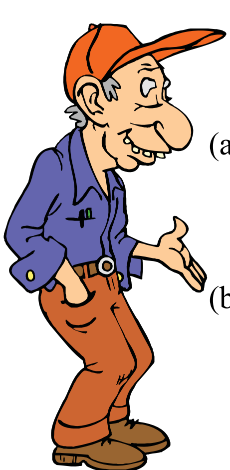
Question 7 A transformation is represented by the matrix $\mathbf{M} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$.

(a) Describe, fully, this transformation.
(b) (i) Calculate $\mathbf{M}^4$.
(ii) Describe, fully, the transformation represented by $\mathbf{M}^4$.

Question 8 A rectangle has vertices A (1 , 0), B (3 , 0), C (3 , 1) and D (1 , 1).
The matrix **M** represents the transformation equivalent to the combination of a reflection in the line $y = x$ followed by a rotation, through 90° clockwise, about the origin.
(a) Calculate the matrix **M**.
(b) The rectangle $ABCD$ is mapped onto the rectangle $A'B'C'D'$ by the transformation represented by the matrix $\mathbf{M}^2$.
Calculate the coordinates of the vertices of $A'B'C'D'$.
(c) What can you say about the transformation represented by $\mathbf{M}^2$?

Question 9 Use axes with $-6 \leq x \leq 3$, $-3 \leq y \leq 2$.

The transformation T is represented by the matrix, $\mathbf{M} = \begin{pmatrix} 0 & -3 \\ 1 & 0 \end{pmatrix}$.



Rectangle R has vertices (0 , 0), (2 , 0), (2 , 1) and (0 , 1).
(a) The transformation T maps R onto R_1 , so that $T(R) = R_1$.
Calculate the coordinates of the vertices of R_1.
Draw and label R and R_1 on your diagram.
Describe a combination of two transformations which is equivalent to T.
(b) R_1 is mapped onto R_2 by the transformation T, so that $T(R_1) = R_2$.
Calculate the coordinates of the vertices of R_2.
Draw and label R_2 on your diagram.
Describe the **single** transformation that maps R directly onto R_2.
Calculate the matrix which represents this single transformation.

Inverse Transformations and Inverse Matrices.

> An **inverse** transformation has the opposite effect to the original transformation.

Example 1 The inverse transformation to the transformation of rotation, through 60° clockwise, about the point A, is the rotation, through 60° anticlockwise, about the point A.

Question 1 Describe the inverse transformations of
- (a) a rotation, about the origin, through 90° anticlockwise,
- (b) an enlargement, centre the origin, with scale factor 2,
- (c) a reflection in the x-axis,
- (d) a reflection in the line $y = x$,
- (e) a one-way stretch, y-axis invariant, scale factor 3,
- (f) a shear, x-axis invariant, mapping (1 , 1) onto (3 , 1).

Which of these *two* transformations are their own inverses?

> When a transformation is represented by the matrix **M**, the inverse transformation is represented by $\mathbf{M}^{-1}$.

Question 2 Each part of this question contains a transformation and the matrix which represents it.
For each part (i) find the inverse matrix,
(ii) decide whether the transformation is its own inverse.

(a) a reflection in the y-axis, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$,

(b) an enlargement, centre the origin, with scale factor -2, $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$,

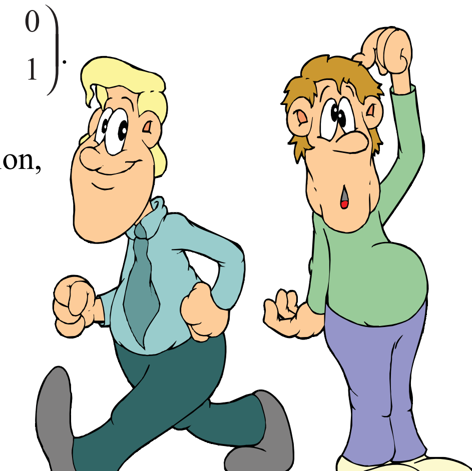
(c) a one-way stretch, x-axis invariant, scale factor 3, $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$,

(d) a rotation, about the origin, through 180°, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$,

(e) a reflection in the line $y = -x$, $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$,

(f) a shear, y-axis invariant, mapping (1 , 1) onto (1 , 2), $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Question 3 For each of the transformations given in Question 1
- (i) write down the matrix which represents the transformation,
- (ii) write down the inverse matrix,
- (iii) find the square of the matrix,
- (iv) decide whether the matrix is its own inverse.



Question 4 For each of the following matrices, find
- (i) the square of the matrix,
- (ii) the inverse matrix.

What do you notice?

(a) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}$ (c) $\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$ (e) $\begin{pmatrix} -1 & 0 \\ -3 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}$

If a matrix M is its own inverse then $\mathbf{M}^2 = \mathbf{I}$.

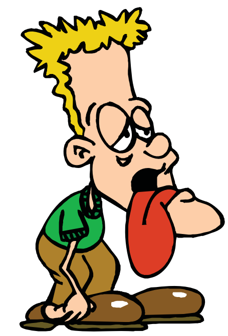
Question 5 Use both x and y axes from 0 to 6.

(a) The square with vertices (0 , 2), (1 , 2), (1 , 3), (0 , 3) is transformed by the matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}.$$

Draw both the square and its image on the same diagram.

(b) The parallelogram with vertices (2 , 4), (2 , 3), (3 , 5), (3 , 6) is transformed by the matrix $\mathbf{Q} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$.



Draw both the parallelogram and its image on another diagram.

(c) (i) What do you notice about your diagrams?

(ii) What do you deduce about the matrices **P** and **Q**?

(iii) Find **PQ** and **QP**.

Question 6 Use both x and y axes from 0 to 6.

(a) The parallelogram with vertices (2 , 2), (3 , 4), (5 , 6), (4 , 4) is transformed by the matrix $\mathbf{M} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$.

Draw both the parallelogram and its image on the same diagram.

(b) The rectangle with vertices (0 , 2), (1 , 2), (1 , 4), (0 , 4) is transformed by the matrix $\mathbf{N} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$.

Draw both the rectangle and its image on another diagram.

(c) (i) What do you notice about your diagrams?

(ii) What do you deduce about the matrices **M** and **N**?

(iii) Find **MN** and **NM**.

Question 7 The matrix $\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ has an inverse $\mathbf{P}^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$.

A transformation T is represented by the matrix **P**.

(a) Find the image of the point A (-2 , 3) under the transformation T.

(b) The transformation T maps the point B onto the point (5 , -1).
Find the coordinates of B.

Question 8 The point P maps onto its image P’ under the transformation which is represented by the matrix $\mathbf{M} = \begin{pmatrix} 4 & 3 \\ 2 & 2 \end{pmatrix}$.

P’ maps back onto P under the inverse transformation.
Find

(a) the inverse matrix $\mathbf{M}^{-1}$,

(b) P’ , when P is (3 , 0),

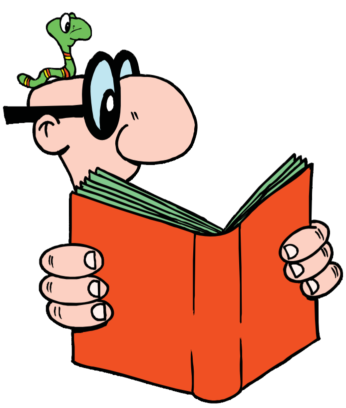
(c) P’ , when P is (-2 , 3),

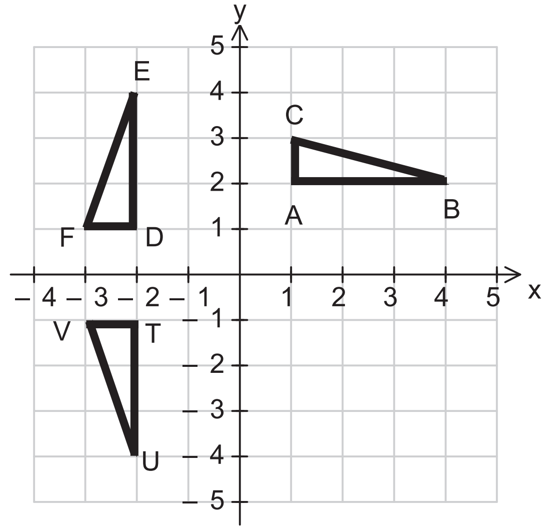
(d) P , when P’ is (0 , 4),

(e) P , when P’ is (5 , -2).

Further Questions on Matrix Transformations.

Question 1





(a) If triangle LMN is the image of triangle ABC under reflection in the x-axis, write down the coordinates of the vertex N.
Write down the matrix $\mathbf{P}$ which represents this transformation.

(b) Describe, **completely**, the **single** transformation that maps triangle ABC onto triangle DEF.
Write down the matrix $\mathbf{Q}$ which represents this transformation.

(c) Write down the matrix $\mathbf{R}$ which represents the transformation that maps triangle ABC directly onto triangle TUV.

(d) Write down an equation connecting $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$.

Question 2 Use axes with $-2 \le x \le 3$, $0 \le y \le 4$.

The transformation T is represented by the matrix, $\mathbf{M} = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$.

Rectangle R has vertices (1 , 0), (2 , 0), (1 , 2) and (2 , 2).

(a) The transformation T maps R onto R_1, so that $T(R) = R_1$.
Calculate the coordinates of the vertices of R_1.

(b) Draw and label R and R_1 on your diagram.

(c) R_1 is the reflection of R in a certain line.
Draw this line on your diagram, and write down its equation.

(d) Find (i) $\mathbf{M}^2$, (ii) $\mathbf{M}^{-1}$.

(e) Explain the significance of your answers to (d).

Question 3 A transformation is represented by the matrix $\mathbf{M} = \begin{pmatrix} -4 & -6 \\ 2 & 2 \end{pmatrix}$.

(a) Find the inverse matrix $\mathbf{M}^{-1}$.

(b) Find the matrix $\mathbf{M}^2$.
Show that $\mathbf{M}^2$ can be expressed in the form $k\mathbf{M}^{-1}$, and find the value of k.

(c) Find the matrix $\mathbf{M}^3$.
Describe, **completely**, the transformation which is represented by the matrix $\mathbf{M}^3$.

Question 4 Use axes with $-4 \le x \le 5,\ 0 \le y \le 8$.

The transformation T is represented by the matrix, $\mathbf{M} = \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}$.

Square S has vertices (0 , 0), (1 , 0), (1 , 1) and (0 , 1).

(a) The transformation T maps S onto S'.
Calculate the coordinates of the vertices of S'.
(b) Draw and label S and S' on your diagram.
(c) Find the matrix which represents the transformation that maps S' onto S.
(d) The transformation T is equivalent to an enlargement followed by a rotation.
From your diagram find the scale factor of the enlargement and the angle of rotation.
(e) Find the matrix which represents the rotation.

Question 5 Given $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(a) Describe, **completely**, the transformations represented by
(i) **A**, (ii) **B**, (iii) **AB**, (iv) **BA**.
(b) Find out whether $\mathbf{B}^{-1}\,\mathbf{A}^{-1}$ or $\mathbf{A}^{-1}\,\mathbf{B}^{-1}$ is the inverse of **BA**.
(c) Explain your answer to (b) in terms of geometrical transformations.

Question 6 A reflection in the line $y = -x$ is represented by the matrix **M.**
A rotation, about the origin, through 90° anticlockwise is represented by the matrix **N**.

(a) Find (i) **MN**, (ii) **NM**, (iii) the matrix **X** such that **XNM** = **MN**.
(b) Describe, completely, the transformation represented by the matrix **X**.

Question 7 A transformation maps (1 , 0) onto (0 , -3) and (0 , 1) onto (3 , 0).

(a) Find the matrix which represents this transformation.
(b) Find the image of the point (2 , -1).
(c) Find the coordinates of the point that is mapped onto (-12 , 0).

Question 8 A transformation maps (1 , 0) onto (2 , -1) and (1 , 1) onto (1 , -1).

(a) Find the matrix which represents this transformation.
(b) Find the image of the point (1 , 2).
(c) Find the coordinates of the point that is mapped onto (13 , -5).

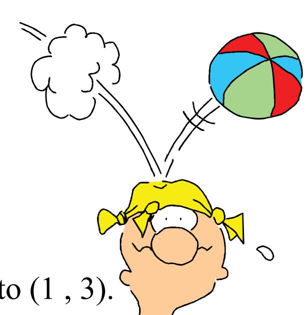
Question 9 A transformation maps (1 , 2) onto (-1 , 3) and (2 , 3) onto (-1 , 5).

(a) Find the matrix which represents this transformation.
(b) Find the image of the point (2 , -1).
(c) Find the coordinates of the point that is mapped onto (12 , 0).

Question 10 The transformation T is represented by the matrix, $\mathbf{M} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) Find the images of the points (1 , 0) and (0 , 1).
(b) Find the coordinates of the point which is mapped onto (8 , 3).
(c) The transformation T is a combination of two transformations.
The first is a rotation, about the origin, through 90° anticlockwise.
Describe the second transformation.

Question 11 The transformation T is represented by the matrix, $\mathbf{M} = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$.



(a) Find the images of the points (1 , 0) and (0 , 1).
(b) Find the coordinates of the point which is mapped onto (1 , -1).
(c) The transformation T is a combination of two transformations.
The second is a shear, with the y-axis invariant, mapping (1 , 1) onto (1 , 3).
Describe the first transformation.

Finding a Transformation Matrix.

Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$ it follows that

the **first column** of a transformation matrix can be obtained by finding the image point of **(1 , 0)**,
the **second column** of a transformation matrix can be obtained by finding the image point of **(0 , 1)**.

Example 1 Find the matrix which represents the transformation reflection in the y-axis.

Under a reflection in the y-axis;

(1 , 0) maps onto (-1 , 0), so the first column is $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$,

(0 , 1) maps onto itself, (0 , 1), so the second column is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

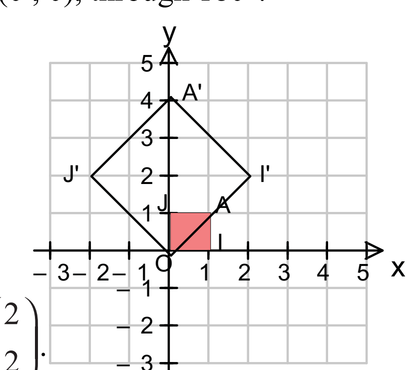
Hence the matrix is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

> The unit square is defined as the square with vertices (0 , 0), (1 , 0), (1 , 1) and (0 , 1).
> The point (1 , 0) is usually labelled I . The point (0 , 1) is usually labelled J.

Question 1 By finding the image points of (1 , 0) and (0 , 1) write down the matrix which represents the transformation

(a) reflection in the line $y = x$, (b) rotation, about (0 , 0), through 180°.

Example 2 The diagram shows the unit square $OIAJ$ and its image $O'I'A'J'$ under a certain transformation.



(a) Find the matrix which represents the transformation.

(b) Check the answer by calculating the image of A (1 , 1).

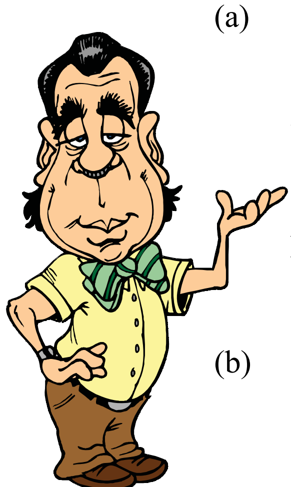
(a) The image of I (1 , 0) is I' (2 , 2), so the first column is $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

The image of J (0 , 1) is J' (-2 , 2), so the second column is $\begin{pmatrix} -2 \\ 2 \end{pmatrix}$.

Hence the matrix is $\begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$.

(b) The image of A (1 , 1) is given by $\begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$.

So A' is (0 , 4) as in the diagram.

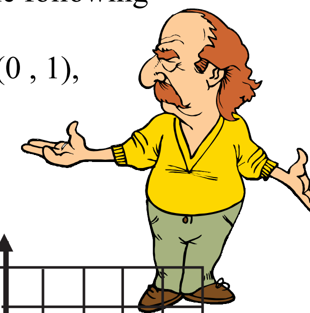


Question 2 For each of the transformations of the unit square shown in the following diagrams

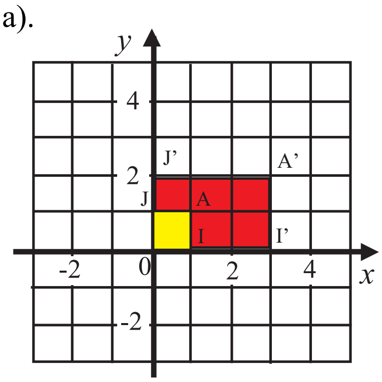
(i) write down the coordinates of the image points of (1 , 0) and (0 , 1),

(ii) write down the matrix which represents the transformation,

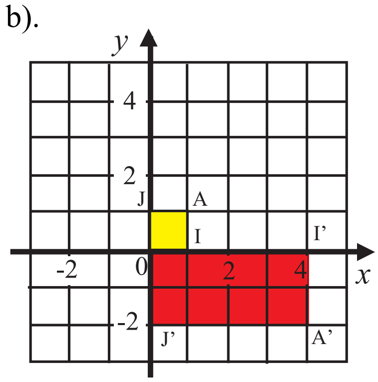
(iii) check your matrix by finding its image of (1 , 1).



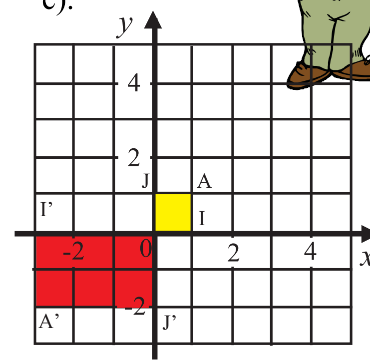
a).



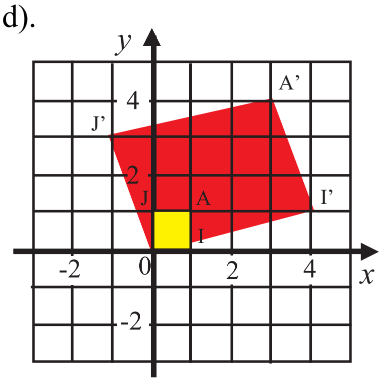
b).



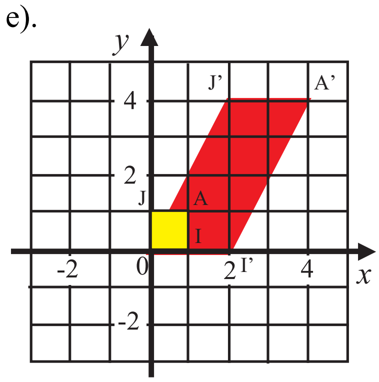
c).



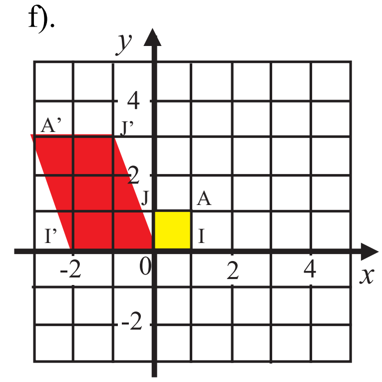
d).



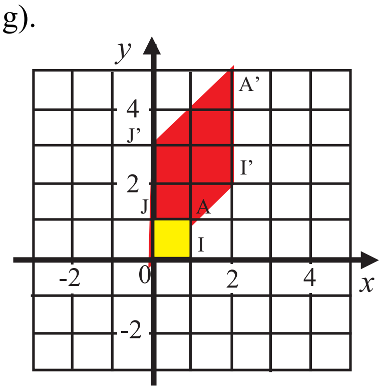
e).



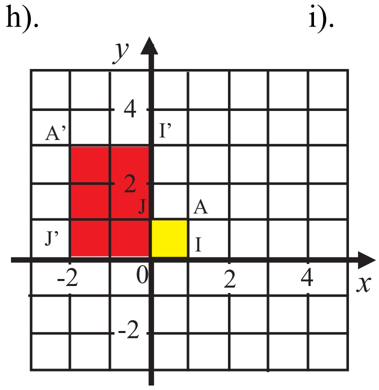
f).



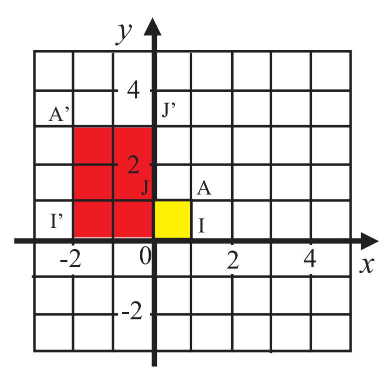
g).



h).



i).



Question 3 For each of the following transformations

(i) draw the unit square *OIAJ* and its image *O'I'A'J'*,

(ii) write down the coordinates of *I'* and *J'* as accurately as you can,

(iii) write down the matrix which represents the transformation.

(a) An enlargement, centre the origin, scale factor -2.

(b) A rotation, about the origin, through 45° clockwise.

(c) A rotation, about the origin, through 45° anticlockwise.

(d) A rotation, about the origin, through 135° anticlockwise.

(e) A rotation, about the origin, through 60° clockwise.

(f) A reflection in the line $y = 2x$.

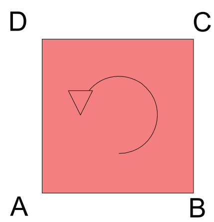
(g) A reflection in the line $y = \frac{1}{2}x$.

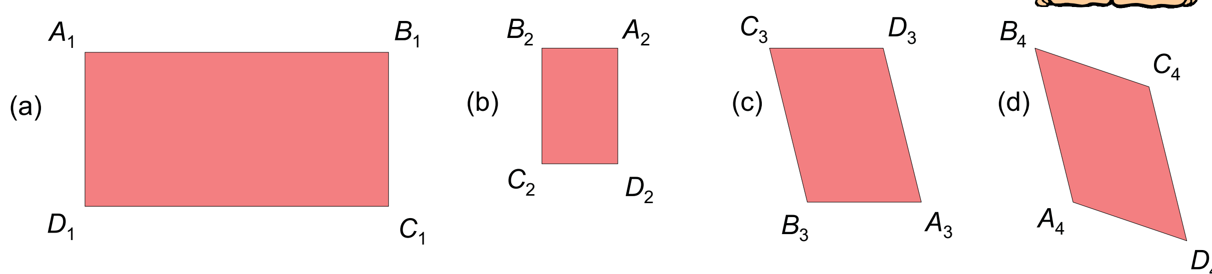
(h) A reflection in the line $y = -$.

(i) A reflection in the line $y = -2x$.

Shape Orientation and Determinants.

Question 1





The diagram shows a square ABCD and four images $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ etc.
Following the corners in the order of the letters gives an anticlockwise direction, as indicated by the arrow.
Follow the corners for each image and state whether the direction is the same as, or opposite to, that of the object.

Question 2 For each of the following transformations state whether the direction round an image shape is the same as, or opposite to, that of the object shape.

(a) a rotation, about the origin, through 90°, clockwise,
(b) a reflection in the *y*-axis,
(c) an enlargement, centre the origin, scale factor 2,
(d) an enlargement, centre the origin, scale factor -2 ,
(e) a one-way stretch, *x*-axis invariant, scale factor 3,
(f) a one-way stretch, *x*-axis invariant, scale factor -3.

Question 3 For each of the transformations in Question 2

(a) write down the matrix which represents it,
(b) calculate the determinant of this matrix.

What is the connection between the sign of the determinant and the direction of the corners of the image as compared to the direction of the corners of the object?

Question 4 For each transformation represented by these matrices, state whether the direction round an image shape is the same as, or different from, the direction round its object.

(a) $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

(e) $\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ (g) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (h) $\begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}$

(i) $\begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}$ (j) $\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$ (k) $\begin{pmatrix} -4 & -3 \\ -3 & 4 \end{pmatrix}$ (l) $\begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix}$

Transformations involving Singular Matrices.

Question 1 Use axes with $0 \le x \le 5$, $0 \le y \le 10$.

(a) Find the image of the square with vertices $A\ (1\ ,\ 1)$, $B\ (2\ ,\ 1)$, $C\ (2\ ,\ 0)$ and $D\ (1\ ,\ 0)$ under the transformation represented by $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$.

(b) Draw both the square and its image on the same diagram.

(c) Describe the image shape.

(d) (i) Write down the area scale factor of the transformation.

(ii) Write down the determinant of the matrix.

(e) Write down the equation of the straight line which passes through the image points.

Question 2 Use axes with $0 \le x \le 4$, $0 \le y \le 2$.

(a) Find the image of the unit square under the transformation represented by $\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$.

(b) Draw both the square and its image on the same diagram.

(c) Describe the image shape.

(d) (i) Write down the area scale factor of the transformation.

(ii) Write down the determinant of the matrix.

(e) Write down the equation of the straight line which passes through the image points.

(f) Find two other points which map onto $(4\ ,\ 0)$.

(g) Find the equation of the line that passes through the points which map onto $(4\ ,\ 0)$.

(h) Find two other points which map onto $(2\ ,\ 0)$.

(i) Find the equation of the line that passes through the points which map onto $(2\ ,\ 0)$.

Question 3 Use axes with $0 \le x \le 2$, $0 \le y \le 2$.

(a) Find the image of the triangle with vertices $A\ (1\ ,\ 0)$, $B\ (1\ ,\ 1)$ and $C\ (1\ ,\ 2)$ under the transformation represented by $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(b) Draw both the triangle and its image on the same diagram.

(c) Describe the image shape.

> If a matrix is **singular** then the transformation it represents can be described as a **collapse**.
> The image points will, in general, lie on a straight line which passes through the origin.
> Many image points will map onto the same point.
> It is not possible to reverse the transformation.
> The matrix which represents the transformation does not have an inverse, since it is a singular matrix.

Question 4 Each of these matrices represents a collapse.
In each case, find the equation of the line which passes through the image points,
either by finding the image of the unit square and drawing it on a diagram
or by using algebra.

(a) $\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & -3 \\ -4 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 6 & -4 \\ 3 & -2 \end{pmatrix}$

(e) $\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 5 \\ 1 & 5 \end{pmatrix}$ (g) $\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$

Area Scale Factors and Determinants.

Question 1 For each of the following matrices

(i) draw and label axes as indicated,
(ii) transform the object shape whose coordinates are given,
(iii) draw both object and image on your axes,
(iv) calculate the area of the object and the area of the image and write down the area scale factor,
(v) write down the determinant of the matrix.

(a) $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, axes : $0 \le x \le 6,\ 0 \le y \le 3$ object : (1 , 0), (2 , 0), (2 , 2), (1 , 2)

(b) $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$, axes : $0 \le x \le 4,\ 0 \le y \le 5$ object : (1 , 0), (3 , 0), (3 , 1), (1 , 1)

(c) $\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$, axes : $0 \le x \le 4,\ 0 \le y \le 5$ object : (1 , 1), (3 , 1), (3 , 2), (1 , 2)

(d) $\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$, axes : $-2 \le x \le 2,\ 0 \le y \le 4$ object : (0 , 0), (1 , 0), (1 , 2), (0 , 2)

(e) $\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$, axes : $0 \le x \le 8,\ 0 \le y \le 3$ object : (0 , 1), (1 , 1), (1 , 2), (0 , 2)

(f) $\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$, axes : $0 \le x \le 8,\ 0 \le y \le 3$ object : (0 , 1), (1 , 1), (1 , 2), (0 , 2)

(g) $\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$, axes : $-3 \le x \le 3,\ 0 \le y \le 10$ object : (1 , 1), (3 , 1), (3 , 3), (1 , 3)

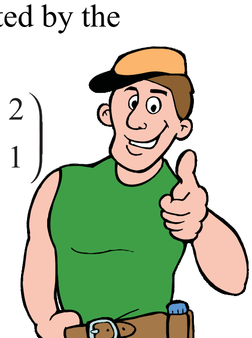
(h) $\begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix}$, axes : $0 \le x \le 8,\ -3 \le y \le 4$ object : (0 , 1), (2 , 1), (2 , 2), (0 , 2)

(i) $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$, axes : $0 \le x \le 8,\ 0 \le y \le 6$ object : (1 , 1), (3 , 1), (3 , 2), (1 , 2)

> The area scale factor of a transformation is equal to the size (irrespective of sign) of the determinant of the matrix which represents the transformation.

Question 2 Find the area scale factor of the transformations which are represented by the following matrices.

(a) $\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & -4 \\ 2 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (e) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (f) $\begin{pmatrix} -2 & 2 \\ 0 & 1 \end{pmatrix}$



Question 3 An object shape has an area of 3 cm^2.
For each transformation represented by the following matrices, find

(i) the area scale factor of the transformation,
(ii) the area of the image shape.

(a) $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} -2 & 4 \\ -3 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 6 & 2 \\ 9 & 3 \end{pmatrix}$ (e) $\begin{pmatrix} 2 & -4 \\ -3 & -1 \end{pmatrix}$ (f) $\begin{pmatrix} -4 & -1 \\ -3 & -2 \end{pmatrix}$

Question 4 Under the transformation represented by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, an object shape maps onto an image which has an area of 5 cm^2. Find the area of the object shape.

Question 5 Under the transformation represented by the matrix $\begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{3} \end{pmatrix}$, an object shape maps onto an image which has an area of 5 cm^2. Find the area of the object shape.

Question 6 Under the transformation represented by the matrix $\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$, an object shape, with an area of 12 cm^2, maps onto a square. Find the length of a side of the image square.

Question 7 Under the transformation represented by the matrix $\begin{pmatrix} 4 & 2 \\ 3 & 2 \end{pmatrix}$, an object square maps onto an image which has an area of 18 cm^2. Find the length of a side of the object square.

Question 8 Under the transformation represented by the matrix $\begin{pmatrix} \frac{1}{4} & 1 \\ 1 & 1 \end{pmatrix}$, an object shape, with an area of 12 cm^2, maps onto a square.
Find the length of a side of the image square.

Question 9 In each of the following cases, find two values of k for which the transformation represented by the matrix produces an image shape which has the same area as the object shape.

(a) $\begin{pmatrix} k & 3 \\ 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -1 \\ k & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 1 \\ -3 & -2k \end{pmatrix}$

Question 10 In each of the following cases, find two values of k for which the transformation represented by the matrix produces an image shape that has an area equal to

(i) twice the area of the object shape,
(ii) half the area of the object shape.

(a) $\begin{pmatrix} k & 3 \\ 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & k \\ 1 & 3 \end{pmatrix}$

Question 11 The tranformation P is represented by the matrix $\mathbf{M} = \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$.

The tranformation Q is represented by the matrix $\mathbf{N} = \begin{pmatrix} 2 & 4 \\ 1 & 0 \end{pmatrix}$.

The transformation P maps a square S onto S_1, so that $P(S) = S_1$.
The transformation Q maps S_1 onto S_2, so that $Q(S_1) = S_2$.

(a) How many times greater is the area of S_2 than the area of S?
(b) Given that the area of S_2 is 450 cm^2, find the length of a diagonal of S.
(c) The transformation R is represented by $\mathbf{N}^{-1}\mathbf{M}$, and $R(S) = S_3$.
How many times greater is the area of S_3 than the area of S?

Level 9/10. Pack 6. Answers.

Page 3. **Matrices 1.**

1 (a) 3 x 2 (b) 3 x 1 (c) 3 x 3 (d) 1 x 1 (e) 1 x 4

Page 4.

2 (a) $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 \\ -5 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 2 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 5 & 2 \end{pmatrix}$ (e) $\begin{pmatrix} -1 \\ 6 \end{pmatrix}$

(f) $\begin{pmatrix} 5 & -1 \end{pmatrix}$ (g) $\begin{pmatrix} 14 \\ 12 \\ 8 \end{pmatrix}$ (h) $\begin{pmatrix} 3 & 4 \\ 4 & -1 \\ 3 & 7 \end{pmatrix}$ (i) $\begin{pmatrix} 3 & 1 \\ -5 & 9 \end{pmatrix}$ (j) $\begin{pmatrix} 7 & 7 & -1 \\ 6 & -6 & 0 \end{pmatrix}$

(k) $\begin{pmatrix} 5 & 6 & -1 \\ 0 & -6 & 0 \end{pmatrix}$

3 (a) $\begin{pmatrix} 3 & 3 \\ -3 & 7 \end{pmatrix}$ (b) $\begin{pmatrix} 7 \\ 0 \end{pmatrix}$ (c) $\begin{pmatrix} 9 & 9 & 9 \end{pmatrix}$ (d) $\begin{pmatrix} 7 & 6 \end{pmatrix}$

Page 5. **Matrices 2.**

4 (a) $\begin{pmatrix} -2 \\ 7 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 6 & 2 & -2 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 6 \end{pmatrix}$ (e) $\begin{pmatrix} 4 \\ 7 \end{pmatrix}$

(f) $\begin{pmatrix} 5 & -5 \end{pmatrix}$ (g) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (h) $\begin{pmatrix} 3 & 0 \\ -6 & 1 \\ 11 & 3 \end{pmatrix}$ (i) $\begin{pmatrix} -4 & 1 \\ 1 & 1 \end{pmatrix}$ (j) $\begin{pmatrix} 7 & 3 & 7 \\ -2 & -2 & -12 \end{pmatrix}$

(k) $\begin{pmatrix} 3 & 10 & 7 \\ -8 & -2 & -12 \end{pmatrix}$

5 (a) $\begin{pmatrix} -3 \\ 9 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 5 \\ -2 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 8 & 4 & 0 \end{pmatrix}$

Page 6.

6 (a) $\begin{pmatrix} -3 \\ 15 \end{pmatrix}$ (b) $\begin{pmatrix} 28 & -42 \\ -35 & 14 \end{pmatrix}$ (c) $\begin{pmatrix} 45 & -25 \end{pmatrix}$ (d) $\begin{pmatrix} 8 & -16 & 24 \\ -36 & 20 & -12 \end{pmatrix}$

(e) $\begin{pmatrix} -3 & 9 \\ 12 & 18 \end{pmatrix}$ (f) $\begin{pmatrix} -2 & 6 \\ 8 & 12 \end{pmatrix}$ (g) $\begin{pmatrix} 4 & 8 \\ 6 & -2 \end{pmatrix}$ (h) $\begin{pmatrix} -9 & 27 \\ 36 & 54 \end{pmatrix}$

7 (a) $\begin{pmatrix} 7 \\ -8 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & 1 \\ 4 & -1 \end{pmatrix}$ (c) $\begin{pmatrix} 12 & 6 & -4 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 14 \end{pmatrix}$ (e) $\begin{pmatrix} 9 \\ 11 \end{pmatrix}$

(f) $\begin{pmatrix} 13 & -15 \end{pmatrix}$

Page 7. **Matrices 3.**

8 (a) $\begin{pmatrix} 14 & 16 & 13 \end{pmatrix}$ (b) The favourite colours of all the children. (c) blue

Page 8.

9 (a)(i) $\begin{pmatrix} 58 & 10 & 22 \\ 137 & 25 & 13 \end{pmatrix}$ (ii) The numbers of cars, buses and lorries travelling up and down the road during the one hour Bill is recording. (iii) up the road

(b) $\mathbf{B} - \mathbf{A} = \begin{pmatrix} 8 & 0 & 2 \\ 13 & 3 & -1 \end{pmatrix}$

10 (a) $\begin{pmatrix} 40 & 85 & 26 \end{pmatrix}$ (b) peas 3; beans 1; carrots 2.

11 (a) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 6 & 2 \\ 0 & 2 & 4 \end{pmatrix}$ (c) Correct diagram

(d) They are congruent. New triangle is triangle *ABC* translated through $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

(e) They are similar. New triangle is triangle *ABC* enlarged, SF 2, Centre the origin.

Page 9. Matrices 4.

12 (a) 2 x 3 (b) 1 x 3 (c) 2 x 2 (d) 3 x 1 (e) 1 x 2 (f) 3 x 4 (g) 3 x 3 (h) 1 x 1 (i) 3 x 2 (j) 2 x 1

13 (a) $\begin{pmatrix} -4 \\ 17 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 9 \end{pmatrix}$ (c) $\begin{pmatrix} 7 \\ 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 3 \\ -9 & 11 \end{pmatrix}$ (e) $\begin{pmatrix} -2 & 2 \\ 4 & -4 \end{pmatrix}$

(f) $\begin{pmatrix} 8 & -7 \end{pmatrix}$ (g) $\begin{pmatrix} 19 \\ -11 \end{pmatrix}$ (h) $\begin{pmatrix} 9 \\ 16 \end{pmatrix}$ (i) $\begin{pmatrix} 0 \\ -11 \end{pmatrix}$ (j) $\begin{pmatrix} 8 & 2 \end{pmatrix}$

(k) $\begin{pmatrix} 7 & 4 \end{pmatrix}$ (l) $\begin{pmatrix} 24 & -21 \end{pmatrix}$ (m) $\begin{pmatrix} -3 & 4 \\ 1 & 6 \end{pmatrix}$ (n) $\begin{pmatrix} 2 & 9 \\ -12 & 2 \end{pmatrix}$ (o) $\begin{pmatrix} 7 & 0 & 5 \\ 7 & -3 & 1 \end{pmatrix}$

(p) $\begin{pmatrix} -7 & -14 & 18 \\ 7 & 6 & -16 \end{pmatrix}$ (q) $\begin{pmatrix} 13 & 9 \\ -11 & 17 \end{pmatrix}$ (r) $\begin{pmatrix} 7 & 12 \\ -1 & 12 \end{pmatrix}$ (s) $\begin{pmatrix} -11 & -5 \\ -5 & -3 \end{pmatrix}$

(t) $\begin{pmatrix} -13 & -4 \\ 3 & 0 \end{pmatrix}$

14 (a) $\begin{pmatrix} 9 \\ -3 \end{pmatrix}$ (b) $\begin{pmatrix} -1 \\ -3 \end{pmatrix}$ (c) $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$ (d) $\begin{pmatrix} 6 \\ 1 \end{pmatrix}$ (e) $\begin{pmatrix} 1 \\ -7 \end{pmatrix}$ (f) $\begin{pmatrix} 5 \\ 8 \end{pmatrix}$

(g) $\begin{pmatrix} -9 \\ 9 \end{pmatrix}$ (h) $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$ (i) $\begin{pmatrix} -12 \\ -13 \end{pmatrix}$

15 (a) $\begin{pmatrix} 4 & 6 \\ 8 & 10 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & -4 \\ -1 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & -5 \\ 2 & 7 \end{pmatrix}$ (e) $\begin{pmatrix} 2 & 4 \\ -5 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} 0 & 7 \\ 11 & 2 \end{pmatrix}$ (g) $\begin{pmatrix} 4 & 18 \\ 11 & 7 \end{pmatrix}$ (h) $\begin{pmatrix} 1 & 4 \\ -2 & 0 \end{pmatrix}$ (i) $\begin{pmatrix} 8 & 11 \\ -12 & 9 \end{pmatrix}$

Page 10

16 (a) NP (b) NP (c) $\begin{pmatrix}1 & -5\\ 1 & 1\end{pmatrix}$ (d) $\begin{pmatrix}0 & 14\end{pmatrix}$ (e) NP (f) $\begin{pmatrix}12\\ 0\end{pmatrix}$

(g) $\begin{pmatrix}2\\ 3\end{pmatrix}$ (h) NP (i) NP (j) NP (k) $\begin{pmatrix}-4 & -1\end{pmatrix}$ (l) NP (m) $\begin{pmatrix}\frac{1}{2} & 2\\ 3\frac{1}{2} & -1\end{pmatrix}$

17 (a) $\begin{pmatrix}3\\ 5\end{pmatrix}$ (b) $\begin{pmatrix}7\\ 7\end{pmatrix}$ (c) $\begin{pmatrix}-3\\ 7\end{pmatrix}$ (d) $\begin{pmatrix}-2\\ 3\end{pmatrix}$ (e) $\begin{pmatrix}\frac{1}{6}\\ \frac{2}{3}\end{pmatrix}$ (f) $\begin{pmatrix}5\\ -1\end{pmatrix}$ (g) $\begin{pmatrix}-5\\ -1\end{pmatrix}$

(h) $\begin{pmatrix}1\\ -1\frac{1}{2}\end{pmatrix}$ (i) $\begin{pmatrix}6\\ -4\end{pmatrix}$ (j) $\begin{pmatrix}1\\ 5\end{pmatrix}$ (k) $\begin{pmatrix}1\\ -\frac{2}{3}\end{pmatrix}$ (l) $\begin{pmatrix}-2\\ 3\end{pmatrix}$ (m) $\begin{pmatrix}-12\\ -7\end{pmatrix}$ (n) $\begin{pmatrix}12\\ -5\end{pmatrix}$

(o) $\begin{pmatrix}0\\ 23\end{pmatrix}$

18 (a) x = 1/4 (b) x = 4 y = -2 (c) x = -2 1/2 y = 2
(d) x = -2 y = -2 (e) x = 3 y = 6 (f) x = 1/3 y = 6
(g) x = 5 y = 2 (h) x = 8 y = 2 (i) x = 0 y = -6
(j) x = -3 y = -1/3 (k) x = 5 y = -2 (l) x = 3 y = 6
(m) x = 2 y = 1 (n) x = 3 y = -1 (o) x = 2 y = -1

19 $\mathbf{S} - 4\mathbf{N} = \begin{pmatrix}422 & 160 & 408\end{pmatrix}$.

Page 11. Matrix Multiplication 1.

1 (a) (43) (b) (22) (c) (-1) (d) (22) (e) (2)
(f) (-22) (g) (53) (h) (53) (i) (-17)

2 (a) 3 (b) 8 (c) 7 (d) 8 (e) 3
(f) 8 (g) -4 (h) 5 or -5 (i) 1 or -6

Page 12.

3 (580) giving \$5.80 4 (25.05) giving \$25.05
5 (a) (395) giving 395 kg (b) 19.75 kg
6 (a) (225) giving 225 km (b) 45 km/h

Page 13. Matrix Multiplication 2.

7 (a) $\begin{pmatrix}16\\ 30\end{pmatrix}$ (b) $\begin{pmatrix}32\\ 26\end{pmatrix}$ (c) $\begin{pmatrix}30\\ 16\end{pmatrix}$ (d) $\begin{pmatrix}26\\ 32\end{pmatrix}$ (e) $\begin{pmatrix}-13\\ 34\end{pmatrix}$ (f) $\begin{pmatrix}-17\\ -26\end{pmatrix}$

(g) $\begin{pmatrix}28\\ 29\end{pmatrix}$ (h) $\begin{pmatrix}24\\ 27\end{pmatrix}$ (i) $\begin{pmatrix}27\\ 24\end{pmatrix}$ (j) $\begin{pmatrix}58\\ 13\end{pmatrix}$ (k) $\begin{pmatrix}22\\ 11\end{pmatrix}$ (l) $\begin{pmatrix}-9\\ 41\end{pmatrix}$

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8 (a) $\begin{pmatrix}13 & 34\end{pmatrix}$ (b) $\begin{pmatrix}17 & 26\end{pmatrix}$ (c) $\begin{pmatrix}13 & 34\end{pmatrix}$ (d) $\begin{pmatrix}17 & 26\end{pmatrix}$ (e) $\begin{pmatrix}-22 & 20\end{pmatrix}$

(f) $\begin{pmatrix}-22 & -20\end{pmatrix}$ (g) $\begin{pmatrix}13 & 14\end{pmatrix}$ (h) $\begin{pmatrix}-26 & 32\end{pmatrix}$ (i) $\begin{pmatrix}7 & -20\end{pmatrix}$ (j) $\begin{pmatrix}28 & 29\end{pmatrix}$

(k) $\begin{pmatrix}22 & 55\end{pmatrix}$ (l) $\begin{pmatrix}34 & -4\end{pmatrix}$ (m) $\begin{pmatrix}7 & -9\end{pmatrix}$ (n) $\begin{pmatrix}-9 & 7\end{pmatrix}$ (o) $\begin{pmatrix}33 & -35\end{pmatrix}$

Page 15. Matrix Multiplication 3.

9 (a) (10) (b) $\begin{pmatrix}17 & 10\end{pmatrix}$ (c) $\begin{pmatrix}14 & -20 & 26\end{pmatrix}$ (d) $\begin{pmatrix}-4\\ 5\end{pmatrix}$ (e) $\begin{pmatrix}-4 & -18\\ 5 & 5\end{pmatrix}$

(f) $\begin{pmatrix} 1 & -17 & 7 \\ 4 & 2 & 0 \end{pmatrix}$ (g) (7) (h) $\begin{pmatrix} 7 & -1 \end{pmatrix}$ (i) $\begin{pmatrix} 13 & -1 & -7 \end{pmatrix}$

(j) $\begin{pmatrix} -5 \\ 31 \end{pmatrix}$ (k) $\begin{pmatrix} 3 & -5 \\ 18 & -4 \end{pmatrix}$

10 (a) (i) 1 x 2 (ii) 2 x 1 (iii) 1 x 1 (b) (i) 1 x 2 (ii) 2 x 2 (iii) 1 x 2
(c) (i) 1 x 2 (ii) 2 x 3 (iii) 1 x 3 (d) (i) 2 x 2 (ii) 2 x 1 (iii) 2 x 1
(e) (i) 2 x 2 (ii) 2 x 2 (iii) 2 x 2 (f) (i) 2 x 2 (ii) 2 x 3 (iii) 2 x 3
(g) (i) 1 x 3 (ii) 3 x 1 (iii) 1 x 1 (h) (i) 1 x 3 (ii) 3 x 2 (iii) 1 x 2
(i) (i) 1 x 3 (ii) 3 x 3 (iii) 1 x 3 (j) (i) 2 x 3 (ii) 3 x 1 (iii) 2 x 1
(k) (i) 2 x 3 (ii) 3 x 2 (iii) 2 x 2

Page 16.

11 (a) NP (b) NP (c) 2 x 2 (d) NP (e) 1 x 1 (f) 2 x 2 (g) 1 x 2
(h) NP (i) 1 x 3 (j) NP (k) NP (l) 2 x 1 (m) NP (n) NP
(o) 2 x 3 (p) NP

12 (a) (7) (b) $\begin{pmatrix} 8 & 4 \\ -2 & -1 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & -1 \end{pmatrix}$ (d) NP (e) NP (f) NP (g) NP

(h) $\begin{pmatrix} 6 \\ 5 \end{pmatrix}$ (i) NP (j) NP (k) $\begin{pmatrix} 7 & -10 \\ 7 & 8 \end{pmatrix}$ (l) $\begin{pmatrix} 1 & -16 \\ 7 & 14 \end{pmatrix}$

13 (a) $\begin{pmatrix} 0 & -1 & 8 \\ 4 & -3 & 18 \end{pmatrix}$ (b) NP (c) $\begin{pmatrix} 3 & 8 \\ 9 & 20 \end{pmatrix}$ (d) $\begin{pmatrix} 15 & 22 \\ 6 & 8 \end{pmatrix}$ (e) NP

(f) $\begin{pmatrix} 7 & 10 \\ 15 & 22 \\ -3 & -4 \end{pmatrix}$ (g) NP (h) NP (i) $\begin{pmatrix} 1 & 2 \\ -2 & -7 \end{pmatrix}$ (j) $\begin{pmatrix} 0 & -1 & 8 \\ 4 & -3 & 18 \\ 2 & 0 & -3 \end{pmatrix}$

(k) NP (l) $\begin{pmatrix} 7 & 9 \\ 11 & 14 \\ 7 & 9 \end{pmatrix}$

14 (a) $\begin{pmatrix} 40 \\ 34 \end{pmatrix}$ (b) United, by 6 points

15 (a) $\begin{pmatrix} 580 & 610 \end{pmatrix}$ (b) Shop A, by 30 cents

Page 17. Communtative Matrices.

1 (a)(i) $\begin{pmatrix} 6 & 13 \\ 9 & 27 \end{pmatrix}$ (ii) $\begin{pmatrix} 12 & 9 \\ 23 & 21 \end{pmatrix}$ (b)(i) $\begin{pmatrix} 11 & 14 \\ 14 & 11 \end{pmatrix}$ (ii) $\begin{pmatrix} 11 & 14 \\ 14 & 11 \end{pmatrix}$

(c)(i) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (d)(i) $\begin{pmatrix} 25 & 25 \\ 15 & 15 \end{pmatrix}$ (ii) $\begin{pmatrix} 11 & 11 \\ 29 & 29 \end{pmatrix}$

(e)(i) $\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$ (ii) $\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$ (f)(i) $\begin{pmatrix} -3 & 0 \\ 5 & 4 \end{pmatrix}$ (ii) $\begin{pmatrix} -3 & 0 \\ 1 & 4 \end{pmatrix}$

(g)(i) $\begin{pmatrix} 1 & 0 \\ -7 & 8 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 0 \\ -7 & 8 \end{pmatrix}$ (h)(i) $\begin{pmatrix} 0 & 6 \\ 12 & -6 \end{pmatrix}$ (ii) $\begin{pmatrix} -2 & 8 \\ 10 & -4 \end{pmatrix}$

2 (a) -1 (b) 12 (c) 3 (d) 5 (e) 5 (f) -3 (g) -1 (h) any value

3 (a) $x = -3, y = 2$ (b) $x = 3, y = 0$ (c) $x = 1, y = 3$ (d) $x = 3, y = 1$

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4 (a) $\mathbf{A^2 = AA} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} \mathbf{7} & \mathbf{10} \\ \mathbf{15} & \mathbf{22} \end{pmatrix}$

(b) $\mathbf{A^3 = A^2 A} = \begin{pmatrix} \mathbf{7} & \mathbf{10} \\ \mathbf{15} & \mathbf{22} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} \mathbf{37} & \mathbf{54} \\ \mathbf{81} & \mathbf{118} \end{pmatrix}$

(c) $\mathbf{AA^2} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} \mathbf{7} & \mathbf{10} \\ \mathbf{15} & \mathbf{22} \end{pmatrix} = \begin{pmatrix} \mathbf{37} & \mathbf{54} \\ \mathbf{81} & \mathbf{118} \end{pmatrix}$

(d) $\mathbf{AB} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 4 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{9} & \mathbf{4} \\ \mathbf{23} & \mathbf{12} \end{pmatrix}$

(e) $\mathbf{BA} = \begin{pmatrix} 5 & 4 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} \mathbf{17} & \mathbf{26} \\ \mathbf{2} & \mathbf{4} \end{pmatrix}$

(f) $\mathbf{A - B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 4 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{-4} & \mathbf{-2} \\ \mathbf{1} & \mathbf{4} \end{pmatrix}$

(g) $\mathbf{A(A - B)} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -4 & -2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} -2 & 6 \\ -8 & 10 \end{pmatrix}$

(h) $\mathbf{A^2 - AB} = \begin{pmatrix} \mathbf{7} & \mathbf{10} \\ \mathbf{15} & \mathbf{22} \end{pmatrix} - \begin{pmatrix} \mathbf{9} & \mathbf{4} \\ \mathbf{23} & \mathbf{12} \end{pmatrix} = \begin{pmatrix} \mathbf{-2} & \mathbf{6} \\ \mathbf{-8} & \mathbf{10} \end{pmatrix}$

(i) $\mathbf{A^2 - BA} = \begin{pmatrix} \mathbf{7} & \mathbf{10} \\ \mathbf{15} & \mathbf{22} \end{pmatrix} - \begin{pmatrix} \mathbf{17} & \mathbf{26} \\ \mathbf{2} & \mathbf{4} \end{pmatrix} = \begin{pmatrix} \mathbf{-10} & \mathbf{-16} \\ \mathbf{13} & \mathbf{18} \end{pmatrix}$

(j) $\mathbf{(A - B)A} = \begin{pmatrix} \mathbf{-4} & \mathbf{-2} \\ \mathbf{1} & \mathbf{4} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} \mathbf{-10} & \mathbf{-16} \\ \mathbf{13} & \mathbf{18} \end{pmatrix}$

5 (a) $\begin{pmatrix} 22 & 15 \\ 10 & 7 \end{pmatrix}$ (b) $\begin{pmatrix} 27 & 19 \\ 0 & 8 \end{pmatrix}$ (c) $\begin{pmatrix} 12 & 10 \\ 6 & 4 \end{pmatrix}$ (d) $\begin{pmatrix} 14 & 10 \\ 4 & 2 \end{pmatrix}$ (e) $\begin{pmatrix} -2 & 0 \\ 2 & 2 \end{pmatrix}$

(f) $\begin{pmatrix} 34 & 25 \\ 16 & 11 \end{pmatrix}$ (g) $\begin{pmatrix} -10 & -5 \\ -4 & -3 \end{pmatrix}$ (h) $\begin{pmatrix} 34 & 24 \\ 16 & 10 \end{pmatrix}$ (i) $\begin{pmatrix} 13 & 10 \\ 10 & 3 \end{pmatrix}$ (j) $\begin{pmatrix} 11 & 10 \\ 12 & 5 \end{pmatrix}$

(k) $\begin{pmatrix} 79 & 55 \\ 40 & 24 \end{pmatrix}$ (l) $\begin{pmatrix} 84 & 55 \\ 30 & 29 \end{pmatrix}$ (m) $\begin{pmatrix} 55 & 40 \\ 22 & 19 \end{pmatrix}$ (n) $\begin{pmatrix} 57 & 40 \\ 20 & 17 \end{pmatrix}$

Page 19. Identity and Zero Matrices.

1 (a) (i) and (ii) $\begin{pmatrix} 5 & 3 \\ 2 & 4 \end{pmatrix}$ (b) (i) and (ii) $\begin{pmatrix} 7 & 2 \\ 5 & 6 \end{pmatrix}$ (c) (i) and (ii) $\begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix}$

(d) (i) and (ii) $\begin{pmatrix} 9 & 8 & 7 \\ 4 & 5 & 6 \\ 3 & 2 & 1 \end{pmatrix}$

2 (a) 3x3 (b) 2x2

3 (a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ (e) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

4 (a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (b)(i) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (iii) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (iv) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

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5 (a) (i) and (ii) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (b) (i) and (ii) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (c) (i) and (ii) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

6 (b) Either $x = 0$, or $y = 0$, or both x and y equal zero. No

7 (a) -4 (b) 6 (c) 5 (d) 6

8 (a) Yes (b) No. See, e.g., Q7 (a) and (b) (c) Yes (d) No. Any **A** and **B** where **AB**≠**BA**. (e) Yes (f) No. See, e.g., Q7 (c) and (d) (g) No. **B = 0**

Page 21. Multiple Matrix Multiplication.

1 (a) No (b) Yes 1x3 (c) Yes 2x3 (d) No (e) No (f) Yes 1x1 (g) No (h) No (i) Yes 2x3 (j) Yes 2x2

2 (a) (-26) (b) $\begin{pmatrix} -22 \\ 11 \end{pmatrix}$ (c) $\begin{pmatrix} -4 & -6 \end{pmatrix}$ (d) $\begin{pmatrix} 11 & -4 \\ 22 & -8 \end{pmatrix}$

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3 (a) (i) $\mathbf{PQ} = \begin{pmatrix} 38 & 40 \\ 41 & 38 \\ 45 & 40 \end{pmatrix}$

(ii) The numbers of doors and windows needed at each site.

(b) (i) $\mathbf{QR} = \begin{pmatrix} 3750 \\ 2550 \end{pmatrix}$

(ii) The total cost of the doors and windows at each type of house.

(c) (i) $\mathbf{PQR} = \begin{pmatrix} 17700 \\ 17550 \\ 18750 \end{pmatrix}$

(ii) The total cost of the doors and windows at each site.

4 (a) $\mathbf{XP} = \begin{pmatrix} -2 & -3 & -1 \\ 0 & 1 & 2 \end{pmatrix}$ (b) $\mathbf{YP} = \begin{pmatrix} 4 & 6 & 2 \\ 0 & 2 & 4 \end{pmatrix}$ (c) On a diagram

(d) It is a reflection in the y-axis. (e) It is an enlargement, scale factor 2, centre (0 , 0).

(f) $\mathbf{XYP} = \begin{pmatrix} -4 & -6 & -2 \\ 0 & 2 & 4 \end{pmatrix}$ (g) $\mathbf{YXP} = \begin{pmatrix} -4 & -6 & -2 \\ 0 & 2 & 4 \end{pmatrix}$

(h) They are the same triangle.

Page 23. Inverse of Square Matrices of Order 2 (1).

1 Answers should verify that **PQ = I** and **QP = I**.

2 (a) $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} -3 & -2 \\ -4 & -3 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 9 & 5 \\ 7 & 4 \end{pmatrix}$ (g) $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ (h) $\begin{pmatrix} -5 & -7 \\ 3 & 4 \end{pmatrix}$

3 (a) $x = 10$ $y = 11$ (b) $x = -9$ $y = 16$ (c) $x = 6$ $y = 5$ (d) $x = 3$ $y = 4$
(e) $x = -3$ $y = -7$ (f) $x = 6$ $y = 5$

Page 24.

Determinants of square matrices of order 2

1 (a) -2 (b) 0 (c) -10 (d) 14 (e) -24 (f) 6

2 (a) 7 (b) $\frac{5}{8}$ (c) $-5\frac{1}{2}$ (d) 3 (e) ± 2 (f) $-\frac{1}{2}$

3 (a) 4 (b) 9 (c) 4 (d) 9 (e) k^2

4 (a) (i) 7 (ii) -2 (iii) $\begin{pmatrix} 16 & 11 \\ 26 & 17 \end{pmatrix}$ (iv) $\begin{pmatrix} 24 & 23 \\ 10 & 9 \end{pmatrix}$ (v) -14 (vi) -14

(b) det(**PQ**) = det(**QP**) = (det**P**) x (det**Q**)

5 (a) (i) 2 (ii) $\frac{1}{2}$ (iii) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (iv) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) **A** and **B** are inverses of each other.

(c) $\det \mathbf{A} = \dfrac{1}{\det\mathbf{B}}$ (or $\det \mathbf{B} = \dfrac{1}{\det\mathbf{A}}$)

6 (a) (i) 20 (ii) 20 (iii) $\begin{pmatrix} 20 & 0 \\ 0 & 20 \end{pmatrix}$ (iv) $\begin{pmatrix} 20 & 0 \\ 0 & 20 \end{pmatrix}$ (b) $k = 20$

Page 25. Inverse of Square Matrices of Order 2 (2).

1 (a) $\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$ (b) No inverse (c) $\begin{pmatrix} 0.1 & -0.3 \\ 0.2 & 0.4 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$

(e) $\begin{pmatrix} 0 & -\frac{1}{6} \\ -\frac{1}{4} & -\frac{1}{12} \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}$ (g) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (h) No inverse (i) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(j) $\begin{pmatrix} 0.6 & -0.2 \\ -0.8 & 0.6 \end{pmatrix}$ (k) $\begin{pmatrix} 0.4 & -0.2 \\ 0.2 & -0.6 \end{pmatrix}$ (l) No inverse

2 (a) 6 (b) $1\frac{1}{5}$ (c) -6 (d) $2\frac{1}{2}$ (e) ± 3 (f) 0

3 (a) $\begin{pmatrix} 4 & 11 \\ 2 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 7 & 19 \\ 1 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 \end{pmatrix}$ (e) $\begin{pmatrix} 3 & -5\frac{1}{2} \\ -1 & 2 \end{pmatrix}$

(f) $\begin{pmatrix} 1\frac{1}{2} & -9\frac{1}{2} \\ -\frac{1}{2} & 3\frac{1}{2} \end{pmatrix}$ (g) $\begin{pmatrix} 1\frac{1}{2} & -9\frac{1}{2} \\ -\frac{1}{2} & 3\frac{1}{2} \end{pmatrix}$ (h) $\begin{pmatrix} 3 & -5\frac{1}{2} \\ -1 & 2 \end{pmatrix}$. Equal pairs are (e) and (h) ; (f) and (g)

4 (a) $x = 15$ (b) $\begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & -15 \\ -5 & 19 \end{pmatrix}$ (d) $\begin{pmatrix} 4 & -15 \\ -5 & 19 \end{pmatrix}$ (e) They are equal.

5 a = -2, b = 1, $c = 1\frac{1}{2}$ and $d = -\frac{1}{2}$

6 (a) 1 and 5 (b) 1 and $\frac{1}{5}$

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Solving equations with matrices

1 (a) $\begin{pmatrix} 2 & -13 \\ -2 & 16 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & -4 \\ -4 & 6 \end{pmatrix}$ (c) $\begin{pmatrix} 0.9 & 0.5 \\ -0.3 & -0.5 \end{pmatrix}$ (d) $\begin{pmatrix} 0.2 & -0.4 \\ -0.4 & -2.2 \end{pmatrix}$

(e) $\begin{pmatrix} 3 & 0 \\ -9 & 6 \end{pmatrix}$ (f) $\begin{pmatrix} -6.5 & 9.5 \\ 2.5 & -1.5 \end{pmatrix}$

2 (a) x = -2 y = 10 (b) x = 2 y = 3 (c) x = 5 y = -1

Page 27. Miscellaneous Matrix Questions (1)

1 (a) $\begin{pmatrix} 3 \\ 7 \end{pmatrix}$ (b) $\begin{pmatrix} -3 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ (d) $\begin{pmatrix} 21 \\ -4 \end{pmatrix}$ (e) $\begin{pmatrix} 13 \\ -7 \end{pmatrix}$ (f) $\begin{pmatrix} -3 \\ -6 \end{pmatrix}$ (g) $\begin{pmatrix} -3 & -1 \\ -12 & 9 \end{pmatrix}$

(h) $\begin{pmatrix} -17 & -11 \\ 7 & -10 \end{pmatrix}$ (i) $\begin{pmatrix} 2 & -5 \\ 0 & 4 \end{pmatrix}$ (j) $\begin{pmatrix} 8 & -6 \\ 0 & -2 \end{pmatrix}$ (k) $\begin{pmatrix} 3 & -4 \\ 3 & -8 \end{pmatrix}$

2 (a) $\begin{pmatrix} 0 & 6 \\ 10 & -2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -3 \\ -1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} -1 & 2 \\ 0 & -4 \end{pmatrix}$ (d) $\begin{pmatrix} -2 & 7 \\ 5 & -9 \end{pmatrix}$ (e) $\begin{pmatrix} -3 & 4 \\ 2 & 4 \end{pmatrix}$

(f) $\begin{pmatrix} 5 & -5 \\ 3 & 3 \end{pmatrix}$ (g) $\begin{pmatrix} 3 & 0 \\ 10 & 10 \end{pmatrix}$ (h) $\begin{pmatrix} -1 & 1 \\ 1 & 4 \end{pmatrix}$ (i) $\begin{pmatrix} -10 & 17 \\ 11 & 7 \end{pmatrix}$ (j) $\begin{pmatrix} 0 & 12 \\ 5 & -14 \end{pmatrix}$

(k) $\begin{pmatrix} -24 & 0 \\ 48 & -30 \end{pmatrix}$ (l) $\begin{pmatrix} -6 & 12 \\ 22 & -164 \end{pmatrix}$

3 (a) NP (b) NP (c) $\begin{pmatrix} -1 & -3 \\ -1 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 6 \\ 4 \end{pmatrix}$ (e) NP (f) $\begin{pmatrix} -12 & 18 \end{pmatrix}$ (g) $\begin{pmatrix} 7 & -2 \end{pmatrix}$

(h) NP (i) NP (j) NP (k) $\begin{pmatrix} -5 \\ -2 \end{pmatrix}$ (l) NP (m) $\begin{pmatrix} 4 & -1\frac{1}{2} \\ 1 & -1\frac{1}{2} \end{pmatrix}$

4 (a) $\begin{pmatrix} 12 & 24 \\ 6 & 12 \end{pmatrix}$ (b) (24) (c) NP (d) $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ (e) NP (f) NP (g) $\begin{pmatrix} 6 & -3 \end{pmatrix}$

(h) NP (i) NP (j) NP (k) $\begin{pmatrix} 3 & 6 \\ 1 & -2 \end{pmatrix}$ (l) $\begin{pmatrix} 6 & -9 \\ 2 & -5 \end{pmatrix}$

5 (a) (i) 1 (ii) $\begin{pmatrix} 4 & -3 \\ -9 & 7 \end{pmatrix}$ (b) (i) 2 (ii) $\begin{pmatrix} 2 & -1\frac{1}{2} \\ -3 & 2\frac{1}{2} \end{pmatrix}$ (c) (i) -1 (iii) $\begin{pmatrix} -7 & -5 \\ 4 & 3 \end{pmatrix}$

(d) (i) -3 (ii) $\begin{pmatrix} -1\frac{2}{3} & 2 \\ -\frac{2}{3} & 1 \end{pmatrix}$

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6 $p = 5,\ q = 2,\ r = -1,\ s = 22$

7 (a) $x = -3 \quad y = 0$ (b) $x = 2 \quad y = -\frac{1}{2}$ (c) $x = -2 \quad y = 6$

(d) $x = 2 \quad y = -3$ (e) $x = -4.5 \ y = -3$ (f) $x = 8 \quad y = 9$

(g) $x = 3 \quad y = -1$ (h) $x = 10 \quad y = 7$ (i) $x = -1\frac{1}{2} \quad y = 2\frac{1}{2}$

(j) $x = 2 \quad y = 3$ (k) $x = -2 \quad y = 1$ (l) $x = -1 \quad y = 1$

8 **NB** The signs of all six elements in **A** and **B** could be the opposite of the ones shown here.

(a) $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 2 & 5 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} -7 \\ 4 \end{pmatrix}$

(c) $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -6 & 5 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$ (d) $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ -1 & 3 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 25 \\ -17 \end{pmatrix}$

(e) $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 12 \\ 25 \end{pmatrix}$ (f) $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} -12 \\ 34 \end{pmatrix}$

9 (a) $\begin{pmatrix} -1 \\ 6 \end{pmatrix}$ (b) $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$ (c) $\begin{pmatrix} 4 \\ -8 \end{pmatrix}$ (d) $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ (e) $\begin{pmatrix} 5 & -1 \\ 3 & 2 \end{pmatrix}$ (f) $\begin{pmatrix} 3 & 2 \\ -1 & 3 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ (h) $\begin{pmatrix} 1 & -1 \\ -2 & -1 \end{pmatrix}$ (i) $\begin{pmatrix} 3 & 7 \\ -7 & -17 \end{pmatrix}$ (j) $\begin{pmatrix} 5 & 4 \\ 0 & 2 \end{pmatrix}$ (k) $\begin{pmatrix} 4 & 5 \\ 2\frac{1}{2} & 3 \end{pmatrix}$

(l) $\begin{pmatrix} 4 & 5 \\ 2 & 0 \end{pmatrix}$ (m) $\begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}$

10 (a) $\begin{pmatrix} 16 & 18 \\ 0 & 4 \end{pmatrix}$ (b) $-\frac{3}{8}$ (c) -2 (d) 64

11 $p = 2,\ q = -1$

Page 29. Miscellaneous Matrix Questions (2)

1 (a) $\begin{pmatrix} -1 & -1 & -3 \\ 1 & 4 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 & 3 \\ -1 & -4 & -4 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 4 & 4 \\ 1 & 1 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} -1 & -4 & -4 \\ -1 & -1 & -3 \end{pmatrix}$

(e) On diagram

(f) They are congruent. New triangle is triangle *ABC* rotated through 90°, anticlockwise, about (0 , 0).

(g) They are congruent. New triangle is triangle *ABC* rotated through 90°, clockwise, about (0 , 0).

(h) They are identical.

(i) They are congruent. New triangle is triangle *ABC* rotated through 180°, about (0 , 0).

2 (a) $\begin{pmatrix} 76 \\ 68 \end{pmatrix}$. The mean score in each exam.

(b) $\begin{pmatrix} 146 & 142 & 144 \end{pmatrix}$. The total marks scored by each person.

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3 (a) $u = -16,\ v = 6$ (b) $w = 2,\ x = -2,\ y = 3,\ z = 4$ (c) $a = 2,\ b = 5$

(d) $c = -1$, $d = 3$, (e) $e = 1$, $f = -7$ (f) $g = 3$, $h = 2$ (g) $j = 2$, $k = 8$
(h) $m = 1$, $n = -1$ (i) $p = 2$, $q = 4$, $r = 5$, $s = 13$
(j) $x = 4$, $y = 2$, $z = 11$, $t = 11$

4 (b) (i) $\begin{pmatrix} 32 & 0 \\ 0 & 32 \end{pmatrix}$ (ii) $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ (iii) $\begin{pmatrix} \frac{1}{32} & 0 \\ 0 & \frac{1}{32} \end{pmatrix}$

5 (a) (i) $\begin{pmatrix} 5 & 3 \\ 10 & 6 \end{pmatrix}$ (ii) $\begin{pmatrix} 5 & 3 \\ 10 & 6 \end{pmatrix}$ (b) Either $x = 0$, or $y = z$ (or both). No.

6 (a) $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$ (b) $x = 5$, $y = -1$

7 (a) (i) $\begin{pmatrix} 2 & 0 \\ -6 & -7 \end{pmatrix}$ (ii) $\begin{pmatrix} -1 & 6 \\ 3 & -4 \end{pmatrix}$

(b) $\begin{pmatrix} -2\frac{1}{14} & -\frac{6}{7} \\ \frac{3}{14} & \frac{4}{7} \end{pmatrix}$ (c) $\begin{pmatrix} 2\frac{2}{7} & \frac{3}{7} \\ 1\frac{5}{7} & \frac{4}{7} \end{pmatrix}$ (d) $\begin{pmatrix} \frac{2}{7} & \frac{3}{7} \\ \frac{3}{14} & \frac{1}{14} \end{pmatrix}$

8 (a) $x = 1$ or $x = 2$ (b) $y = 1$ or $y = \frac{1}{2}$

9 $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Page 31/32. Matrices and Transformations (Introduction).

1 (a) $A'(4, 2)$ $B'(8, 2)$ $C'(8, 4)$
$O'(0, 0)$ $P'(2, 0)$ $Q'(2, 2)$ $R'(0, 2)$
(c) Enlargement, scale factor 2, centre (0 , 0).
(d) 4 , 4 . Equal

2 (a) $A'(6, 3)$ $B'(12, 3)$ $C'(12, 6)$
$O'(0, 0)$ $P'(3, 0)$ $Q'(3, 3)$ $R'(0, 3)$
(c) Enlargement, scale factor 3, centre (0 , 0).
(d) 9 , 9 . Equal

3 (a) $A'(-1, 2)$ $B'(-1, 4)$ $C'(-2, 4)$
$O'(0, 0)$ $P'(0, 1)$ $Q'(-1, 1)$ $R'(-1, 0)$
(c) Rotation, through 90°, anticlockwise, about (0 , 0).
(d) 1 , 1 . Equal

4 (a) $A'(1, -2)$ $B'(1, -4)$ $C'(2, -4)$
$O'(0, 0)$ $P'(0, -1)$ $Q'(1, -1)$ $R'(1, 0)$
(c) Rotation, through 90°, clockwise, about (0 , 0).
(d) 1 , 1 . Equal

Page 33. Single Matrix Transformations.

1 (a) (i) $A'(2, -1)$ $B'(3, -1)$ $C'(3, -3)$
$O'(0, 0)$ $P'(1, 0)$ $Q'(1, -1)$ $R'(0, -1)$
(iii) Reflection in the x-axis.
(iv) y = 0.
(v) 1, 1 . Equal in size, opposite in sign.
(b) (i) $A'(1, 2)$ $B'(1, 3)$ $C'(3, 3)$
$O'(0, 0)$ $P'(0, 1)$ $Q'(1, 1)$ $R'(1, 0)$

(iii) Reflection in the line $y = x$.
(iv) $y = x$.
(v) 1, 1 . Equal in size, opposite in sign.

(c) (i) A' (-1 , -2) B' (-1 , -3) C' (-3 , -3)
O' (0 , 0) P' (0 , -1) Q' (-1 , -1) R' (-1 , 0)
(iii) Reflection in the line $y = -x$.
(iv) $y = -x$.
(v) 1, 1 . Equal in size, opposite in sign.

(d) (i) A' (2 , 3) B' (3 , 4) C' (3 , 6)
O' (0 , 0) P' (1 , 1) Q' (1 , 2) R' (0 , 1)
(iii) Shear, with the y-axis invariant mapping (1 , 1) to (1 , 2).
(iv) $x = 0$.
(v) 1, 1 . Equal.

(e) (i) A' (3 , 1) B' (4 , 1) C' (6 , 3)
O' (0 , 0) P' (1 , 0) Q' (2 , 1) R' (1 , 1)
(iii) Shear, with the x-axis invariant mapping (1 , 1) to (2 , 1).
(iv) $y = 0$.
(v) 1, 1 . Equal.

(f) (i) A' (2 , 2) B' (3 , 2) C' (3 , 6)
O' (0 , 0) P' (1 , 0) Q' (1 , 2) R' (0 , 2)
(iii) One-way stretch, x-axis invariant scale factor 2.
(iv) $y = 0$.
(v) 2, 2 . Equal.

(g) (i) A' (4 , 1) B' (6 , 1) C' (6 , 3)
O' (0 , 0) P' (2 , 0) Q' (2 , 1) R' (0 , 1)
(iii) One-way stretch, y-axis invariant scale factor 2.
(iv) $x = 0$.
(v) 2, 2 . Equal.

(h) (i) A' (2 , 1) B' (3 , 1) C' (3 , 3)
O' (0 , 0) P' (1 , 0) Q' (1 , 1) R' (0 , 1)
(iii) No change. Object and image shapes are identical.
(iv) All lines!
(v) 1, 1 . Equal.

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2(a) (i) A' $(1 , \frac{1}{2})$ B' $(1\frac{1}{2} , \frac{1}{2})$ C' $(1\frac{1}{2} , 1\frac{1}{2})$
O' (0 , 0) P' $(\frac{1}{2} , 0)$ Q' $(\frac{1}{2}, \frac{1}{2})$ R' $(0 , \frac{1}{2})$
(iii) Enlargement.
(iv) $\frac{1}{4}$, $\frac{1}{4}$. Equal.
(v) Scale factor $\frac{1}{2}$, centre (0 , 0).

(b) (i) A' (-4 , -2) B' (-6 , -2) C' (-6 , -6)
O' (0 , 0) P' (-2 , 0) Q' (-2 , -2) R' (0 , -2)
(iii) Enlargement.
(iv) 4, 4. Equal.
(v) Scale factor -2, centre (0 , 0).

(c) (i) A' (2 , 3) B' (3 , 3) C' (3 , 9)
O' (0 , 0) P' (1 , 0) Q' (1 , 3) R' (0 , 3)
(iii) One-way stretch.
(iv) 3, 3 . Equal.
(v) Parallel to the y-axis : $y = 0$.

(d) (i) $A'(6 , 1)$ $B'(9 , 1)$ $C'(9 , 3)$
$O'(0 , 0)$ $P'(3 , 0)$ $Q'(3 , 1)$ $R'(0 , 1)$
(iii) One-way stretch.
(iv) 3, 3 . Equal.
(v) Parallel to the x-axis : $x = 0$.

(e) (i) $A'(2 , -2)$ $B'(3 , -2)$ $C'(3 , -6)$
$O'(0 , 0)$ $P'(1 , 0)$ $Q'(1 , -2)$ $R'(0 , -2)$
(iii) One-way stretch.
(iv) 2, 2 . Equal in size, opposite in sign.
(v) Parallel to the y-axis : $y = 0$.

3 (i) $A'(1 , 0)$ $B'(1 , -1)$ $C'(3 , 3)$
$O'(0 , 0)$ $P'(0 , -1)$ $Q'(1 , 1)$ $R'(1 , 2)$
(iii) Shear.
(iv) $y = x$.
(v) Parallel to the line $y = x$.

4 (i) $A'(-1 , 4)$ $B'(-1 , 5)$ $C'(-3 , 9)$
$O'(0 , 0)$ $P'(0 , 1)$ $Q'(-1 , 3)$ $R'(-1 , 2)$
(iii) Shear.
(iv) $y = -x$.
(v) Parallel to the line $y = -x$.

Page 35 Multiple Matrix Transformations (1).

Page 36.

1 (a) $P(-1 , 2)$ $Q(-1 , 4)$ $R(-2 , 4)$ (b) $X(-1, -2)$ $Y(-1, -4)$ $Z(-2, -4)$
(c) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ (d) Reflection in the line $y = -x$

2 (a) $A'(1 , -2)$ $B'(1 , -4)$ $C'(2 , -4)$ (b) $A''(-1 , -2)$ $B''(-3 , -4)$ $C''(-2 , -4)$
(c) First: Rotation, through 90°, clockwise, about (0 , 0).
Second: Shear with the x-axis invariant, mapping (2 , 1) to (3 , 1).
(d) $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$

3 (a) $A'(-1 , 0)$ $B'(-3 , 0)$ $C'(-3 , 1)$ $D'(-1 , 1)$
(b) $A''(-1 , 0)$ $B''(-3 , 0)$ $C''(-3 , 2)$ $D''(-1 , 2)$
(c) First: Reflection in the y-axis.
Second: One-way stretch, scale factor 2 with the x-axis invariant.
(d) $\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$

4 (a) $A'(1 , 2)$ $B'(1 , 4)$ $C'(2 , 4)$ (b) $A''(-2 , 1)$ $B''(-4 , 1)$ $C''(-4 , 2)$
(c) First: Reflection in the line $y = x$.
Second: Rotation, through 90°, anticlockwise, about (0 , 0).
(d) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ Reflection in the y-axis.

Page 37. Multiple Matrix Transformations (2).

1 (a) $A'(1 , 0)$ $B'(3 , 0)$ $C'(3 , 2)$ $D'(1 , 2)$
(b) $A''(-1 , 0)$ $B''(-3 , 0)$ $C''(-3 , -2)$ $D''(-1 , -2)$
(c) $A'''(-2 , 0)$ $B'''(-6 , 0)$ $C'''(-6 , -2)$ $D'''(-2 , -2)$
(d) First: One-way stretch, scale factor 2 with the x-axis invariant.
Second: Rotation through 180° about the origin.

Third: One-way stretch, scale factor 2 with the y-axis invariant.

(e) $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ Enlargement, scale factor -2, centre (0 , 0).

2 (a) (1 , 1) (1 , 3) and (2 , 3)
(b) (1 , -1) (1 , -3) and (2 , -3)
(c) (-1 , 1) (-1 , 3) and (-2 , 3)
(d) Rotation through 180° about the origin. $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

3 Rotation, through 90°, clockwise, about (0 , 0). $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

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4 Reflection in the line $y = -x$. $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

5 Shear with the x-axis invariant, mapping (1 , 1) to (2 , 1).

6 Shear with the y-axis invariant, mapping (1 , 1) to (1 , 3).

7 (a) Shear with the x-axis invariant, mapping (1 , 1) to (4 , 1).
(b)(i) $\begin{pmatrix} 1 & 12 \\ 0 & 1 \end{pmatrix}$ (ii) Shear with the x-axis invariant, mapping (1 , 1) to (13 , 1).

8 (a) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (b)) A'(1, 0) B'(3 , 0) C'(3 , 1) D'(1 , 1)
(c) It is the identity transformation. The image is identical to the object.

9 (a) (0 , 0) (0 , 2) (-3 , 2) (-3 , 1)
One-way stretch, scale factor 3 with the x-axis invariant, followed by a rotation, through 90°, anticlockwise, about (0 , 0).
(b) (0 , 0) (-6 , 0) (-6 , -3) (-3 , -3)
Enlargement, scale factor -3, centre (0 , 0).

Page 39. Inverse Transformations and Inverse Matrices.

1 (a) a rotation, about the origin, through 90° clockwise;
(b) an enlargement, centre the origin, with scale factor $\frac{1}{2}$;
(c) a reflection in the x-axis;
(d) a reflection in the line $y = x$;
(e) a one-way stretch, y-axis invariant, scale factor $\frac{1}{3}$;
(f) a shear, x-axis invariant, mapping (1 , 1) onto (-1 , 1).
(c) and (d)

2 (a) (i) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (ii) Yes (b) (i) $\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ (ii) No

(c) (i) $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$ (ii) No (d) (i) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (ii) Yes

(e) (i) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ (ii) Yes (f) (i) $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ (ii) No

3 (a) (i) $\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$ (ii) $\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$ (iii) $\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$ (iv) No

(b) (i) $\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}$ (ii) $\begin{pmatrix}\frac{1}{2} & 0\\0 & \frac{1}{2}\end{pmatrix}$ (iii) $\begin{pmatrix}4 & 0\\0 & 4\end{pmatrix}$ (iv) No

(c) (i) $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ (ii) $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ (iii) $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ (iv) Yes

(d) (i) $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$ (ii) $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$ (iii) $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ (iv) Yes

(e) (i) $\begin{pmatrix}3 & 0\\0 & 1\end{pmatrix}$ (ii) $\begin{pmatrix}\frac{1}{3} & 0\\0 & 1\end{pmatrix}$ (iii) $\begin{pmatrix}9 & 0\\0 & 1\end{pmatrix}$ (iv) No

(f) (i) $\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$ (ii) $\begin{pmatrix}1 & -2\\0 & 1\end{pmatrix}$ (iii) $\begin{pmatrix}1 & 4\\0 & 1\end{pmatrix}$ (iv) No

4 (a) (i) $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ (ii) $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ (b) (i) $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ (ii) $\begin{pmatrix}1 & 3\\0 & -1\end{pmatrix}$

(c) (i) $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ (ii) $\begin{pmatrix}-1 & 2\\0 & 1\end{pmatrix}$ (d) (i) $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ (ii) $\begin{pmatrix}1 & 0\\2 & -1\end{pmatrix}$

(e) (i) $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ (ii) $\begin{pmatrix}-1 & 0\\-3 & 1\end{pmatrix}$ (f) (i) $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ (ii) $\begin{pmatrix}2 & 1\\-3 & -2\end{pmatrix}$

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5 (a) and (b) The parallelogram is the image of the square and vice versa.
(c) (i) They are identical. (ii) They are inverses of each other. (iii) Both equal **I**.

6 (a) and (b) The rectangle is the image of the parallelogram and vice versa.
(c) (i) They are identical. (ii) They are inverses of each other. (iii) Both equal **I**.

7 (a) (4 , 3) (b) (7 , -1)

8 (a) $\begin{pmatrix}1 & -1\frac{1}{2}\\-1 & 2\end{pmatrix}$ (b) (12 , 6) (c) (1 , 2) (d) (-6 , 8) (e) (8 , -9)

Page 41. Further Questions on Matrix Transformations.

1 (a) (1 , -3) $\mathbf{P} = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$.

(b) Rotation, about (0 , 0), through 90° anticlockwise. $\mathbf{Q} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$.

(c) $\mathbf{R} = \begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$ (d) **R = PQ** or **P = QR**.

2 (a) $\left(-\frac{3}{5},\frac{4}{5}\right)$ $\left(-1\frac{1}{5},1\frac{3}{5}\right)$ $(1,2)$ $\left(\frac{2}{5},2\frac{4}{5}\right)$ (c) $y = 2x$ (d) (i) **I** (ii) **M**

(e) (i) Performing the same reflection twice places the image back onto the object .
(ii) A reflection is its own inverse.

3 (a) $\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 12 \\ -4 & -8 \end{pmatrix}$ $k = 8$

(c) $\mathbf{M}^3 = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$ Enlargement, centre (0 , 0), scale factor 8

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4 (a) (0 , 0) (4 , 3) (1 , 7) (-3 , 4) (c) $\begin{pmatrix} \frac{4}{25} & \frac{3}{25} \\ -\frac{3}{25} & \frac{4}{25} \end{pmatrix}$

(d) scale factor 5, angle approx. 37° (e) $\begin{pmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix}$

5 (a)(i) Reflection in the x-axis (ii) Reflection in the line $y = x$
(iii) Rotation, about (0 , 0), through 90° clockwise
(iv) Rotation, about (0 , 0), through 90° anticlockwise
(b) $\mathbf{A}^{-1}\mathbf{B}^{-1}$
(c) Reflection in the x-axis followed by Reflection in the line $y = x$ is equivalent to
Rotation, about (0 , 0), through 90° anticlockwise whose inverse is
Rotation, about (0 , 0), through 90° clockwise, which is equal to **AB**.
Since reflections are self inverse, $\mathbf{AB} = \mathbf{A}^{-1}\mathbf{B}^{-1}$

6 (a) (i) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (iii) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
(b) Rotation, about (0 , 0), through 180°

7 (a) $\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$ (b) (-3 , -6) (c) (0 , -4)

8 (a) $\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$ (b) (0 , -1) (c) (5 , -3)

9 (a) $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ (b) (3 , 1) (c) (6 , -6)

10 (a) (0 , 1) (-2 , 0) (b) (3 , -4)
(c) One-way stretch, y-axis invariant, scale factor 2

11 (a) (1 , 2) (-1 , -1) (b) (-2 , -3)
(c) Shear, x-axis invariant, mapping (1 , 1) onto (0 , 1).

Page 43. Finding a Transformation Matrix

1 (a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

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2 (a) $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$ (c) $\begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix}$ (d) $\begin{pmatrix} 4 & -1 \\ 1 & 3 \end{pmatrix}$ (e) $\begin{pmatrix} 2 & 2 \\ 0 & 4 \end{pmatrix}$

(f) $\begin{pmatrix} -2 & -1 \\ 0 & 3 \end{pmatrix}$ (g) $\begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix}$ (h) $\begin{pmatrix} 0 & -2 \\ 3 & 0 \end{pmatrix}$ (i) $\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$

3 (a) $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ (b) $\begin{pmatrix} 0.71 & 0.71 \\ -0.71 & 0.71 \end{pmatrix}$ (c) $\begin{pmatrix} 0.71 & -0.71 \\ 0.71 & 0.71 \end{pmatrix}$ (d) $\begin{pmatrix} -0.71 & -0.71 \\ 0.71 & -0.71 \end{pmatrix}$

(e) $\begin{pmatrix} 0.5 & 0.87 \\ -0.87 & 0.5 \end{pmatrix}$ (f) $\begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix}$ (g) $\begin{pmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{pmatrix}$ (h) $\begin{pmatrix} 0.6 & -0.8 \\ -0.8 & -0.6 \end{pmatrix}$

(i) $\begin{pmatrix} -0.6 & -0.8 \\ -0.8 & 0.6 \end{pmatrix}$

Page 45. Shape Orientation and Determinants

1 $A_1B_1C_1D_1$ opposite : $A_2B_2C_2D_2$ same : $A_3B_3C_3D_3$ opposite : $A_4B_4C_4D_4$ opposite

2 (a) same (b) opposite (c) same (d) same (e) same (f) opposite

3 (a) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ 1 (b) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ -1 (c) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ 4 (d) $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ 4

(e) $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ 3 (f) $\begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$ -3

Positive sign - same direction; Negative sign - opposite direction

4 (a) same (b) same (c) same (d) same (e) opposite (f) opposite
(g) opposite (h) opposite (i) same (j) same (k) opposite (l) same

Page 46. Transformations involving Singular Matrices

1 (a) $A'(3, 6)$ $B'(5, 10)$ $C'(4, 8)$ $D'(2, 4)$
(c) Straight line (d) (i) 0 (ii) 0 (e) $y = 2x$

2 (a) $O'(0, 0)$ $I'(2, 0)$ $A'(4, 0)$ $J'(2, 0)$
(c) Straight line (d) (i) 0 (ii) 0 (e) $y = 0$
(f) (2 , 0) , (0 , 2) etc (g) $x + y = 2$ (h) $(\frac{1}{2}, \frac{1}{2})$, (-1 , 2) etc (i) $x + y = 1$

3 (a) All (0 , 0) (c) the single point (0 , 0)

4 (a) $y = 2x$ (b) $y = 3x$ (c) $y = -x$ (d) $y = \frac{1}{2}x$ (e) $y = -\frac{1}{2}x$ (f) $y = x$ (g) $x = 0$

Page 47. Area Scale factors and Determinants.

1 (a)(ii) (2 , 0) (4 , 0) (6 , 2) (4 , 2) (iv) 2, 4, 2 (v) 2
(b)(ii) (1 , 1) (3 , 3) (3 , 5) (1 , 3) (iv) 2, 4, 2 (v) 2
(c)(ii) (2 , 2) (2 , 4) (4 , 5) (4 , 3) (iv) 2, 4, 2 (v) -2
(d)(ii) (0 , 0) (-2 , 0) (-2 , 4) (0 , 4) (iv) 2, 8, 4 (v) -4
(e)(ii) (2 , 1) (5 , 1) (7 , 2) (4 , 2) (iv) 1, 3, 3 (v) 3
(f)(ii) (3 , 0) (5 , 1) (8 , 1) (6 , 0) (iv) 1, 3, 3 (v) -3
(g)(ii) (0 , 3) (2 , 5) (0 , 9) (-2 , 7) (iv) 4, 12, 3 (v) 3
(h)(ii) (2 , 2) (6 , -2) (8 , 0) (4 , 4) (iv) 2, 16, 8 (v) 8
(i)(ii) (1 , 1) (5 , 5) (7 , 4) (5 , 0) (iv) 2, 10, 5 (v) -5

2 (a) 5 (b) 5 (c) 11 (d) 1 (e) 1 (f) 2

3 (a)(i) 2 (ii) 6 cm^2 (b)(i) 1 (ii) 3 cm^2 (c)(i) 10 (ii) 30 cm^2
(d)(i) 0 (ii) 0 cm^2 (e)(i) 14 (ii) 42 cm^2 (f)(i) 5 (ii) 15 cm^2

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4 1.25 cm^2 5 15 cm^2 6 6 cm 7 3 cm 8 3 cm

9 (a) $k = 2$ or 1 (b) $k = -2$ or -4 (c) $k = \frac{2}{3}$ or $\frac{1}{3}$

10 (a)(i) $k = 2\frac{1}{2}$ or $\frac{1}{2}$ (ii) $k = 1\frac{3}{4}$ or $1\frac{1}{4}$
(b)(i) $k = -1$ or -5 (ii) $k = -3\frac{1}{2}$ or $-2\frac{1}{2}$

11 (a) 100 (b) 3 cm (c) $6\frac{1}{4}$